Taking Perturbation to the Accuracy Frontier: A Hybrid of Local and Global Solutions

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Outline

1. Introduction
2. Presentation of the hybrid method
3. Assessing hybrid solutions: a multi-country RBC model
4. Conclusion

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Perturbation versus global methods

Perturbation methods

- Compute approximated solutions using Taylor expansions of optimality conditions around steady state
- Pros: low computational expense, even with high dimensional state space
- Cons: accuracy decreases substantially for state values far from the steady state

Global methods

- Compute solution on large domains; approximate using a finite dimensional functional space
- Arbitrary accuracy level can be achieved
- Pros and cons: precisely the opposite of perturbation

Comparing accuracy of perturbation and global methods

- Several papers in the litterature compare the accuracy of various solution methods, such as Aruoba et al. (JEDC, 2006)
- Last such project: second 2011 issue of the JEDC
- Benchmark model: multi-country RBC model with capital adjustment cost and heterogeneity accross countries
- Accuracy measurement device: normalized Euler errors
- Compares 6 methods: 2 perturbation, 4 global
- Pertubation is noticeably faster, especially for high heterogeneity
- But it is much less accurate:
  - accuracy decreases noticeably as one moves away from the steady state (contrary to global methods)
  - on the ergodic set, has maximum errors larger than those of global methods by several orders of magnitude

The hybrid method: summary of idea and results

- Idea of the hybrid method: start from perturbation solution and improve upon it using global solution techniques
- Extends the available choices in the accuracy/computing cost tradeoff space
- More precisely:
  - solve for some policy functions locally (using standard perturbation)
  - solve for the remaining policy functions globally (using closed-form expressions or a numerical solver)
- Many possible hybrid solutions for a given problem
- In the context of the JEDC 2011 comparison project, with a specific hybrid solution, we obtain a solution more accurate than any other, for a low computing cost

Related literature

- Our generic presentation of the hybrid method encompasses some particular cases studied in the literature
- Dotsey and Mao (JME, 1992):
  - RBC model with labor and production taxes
  - compare linearization with a specific hybrid (capital and labor from perturbation, investment and consumption solved analytically)
  - none of the two methods strictly dominates the other
- Maliar et al. (JEDC, 2011):
  - model from the JEDC 2011 comparison project
  - hybrid method: combine log-linearization for capital with nonlinear solver for consumption and labor
  - the hybrid is about 10 times more accurate than the plain log-linearization

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Studied class of problem (1/2)

$$E_t\left[H\left(\mathbf{x}_t, \mathbf{z}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{z}_{t+1}, \mathbf{y}_{t+1}\right)\right] = \mathbf{0} \tag{1}$$

$$G\left(\mathbf{x}_t, \mathbf{z}_t, \mathbf{y}_t, \mathbf{x}_{t+1}\right) = \mathbf{0} \tag{2}$$

$$\mathbf{z}_{t+1} = \Phi \mathbf{z}_t + \varepsilon_{t+1}$$

where:

- $\mathbf{x}_t \in \mathbb{R}^{n_x}$: endogenous state variables (*e.g.*, capital)
- $\mathbf{z}_t \in \mathbb{R}^{n_z}$: exogenous state (random) variables (*e.g.*, productivity)
- $\mathbf{y}_t \in \mathbb{R}^{n_y}$: control variables (*e.g.*, consumption, labor) and other variables (*e.g.*, prices, Lagrange multipliers) known at t
- $\varepsilon_{t+1} \sim \mathcal{N}\left(\mathbf{0}, \Sigma\right)$
- (1): *inter-temporal choice conditions* (have conditional expectations)
- (2): *intra-temporal choice conditions* (only variables known at t)

Studied class of problem (2/2)

- A solution to the problem is defined as a policy (or decision) function:

$$\Psi : (\mathbf{x}_t, \mathbf{z}_t) \rightarrow (\mathbf{x}_{t+1}, \mathbf{y}_t)$$

  such that all optimality conditions are verified in the relevant region of the state space.
- Note that the number of policy functions is equal to the number of optimality conditions:

$$n \equiv n_x + n_y = n_G + n_H$$

Standard perturbation technique

- Use a Taylor expansion at order p of the optimality conditions, around the steady state
- We denote $\widehat{\Psi}(\mathbf{x}_t, \mathbf{z}_t)$ the approximate policy function delivered by the perturbation method
- As shown by Judd and Guu (1993) and Kollman et al. (JEDC, 2011):
  - accuracy is good near the steady state, but rapidly decreases away from it
  - accuracy on the ergodic state is not sufficient for many economic applications

Constructing a hybrid solution

- Step 1
  - compute a standard perturbation solution $\widehat{\Psi}$
  - partition the n policy functions in 2 groups of sizes n^1 and n^2: $\widehat{\Psi}(\mathbf{x}_t, \mathbf{z}_t) \equiv \left\{\widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t), \widehat{\Psi}^2(\mathbf{x}_t, \mathbf{z}_t)\right\}$
  - discard $\widehat{\Psi}^2$
- Step 2
  - partition the system of n optimality conditions into two sub-systems of sizes n^1 and n^2
  - the sub-system with n^2 equation should identify n^2 policy functions $\Psi^2(\mathbf{x}_t, \mathbf{z}_t)$ uniquely if $\Psi^1(\mathbf{x}_t, \mathbf{z}_t)$ is given
- Step 3
  - given $\widehat{\Psi}^1$ chosen in Step 1, construct (analytically or with a numerical solver) the n^2 policy functions $\widetilde{\Psi}^2$ that satisfy the n^2 equations chosen in Step 2
  - the hybrid solution is: $\widetilde{\Psi}(\mathbf{x}_t, \mathbf{z}_t) \equiv \left\{\widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t), \widetilde{\Psi}^2\left(\mathbf{x}_t, \mathbf{z}_t; \widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t)\right)\right\}$

Choosing a hybrid solution (1/2)

- There are many ways of constructing a hybrid solution for a given model
- Two degrees of freedom:
  - which perturbation policy functions to keep
  - which optimality conditions to use for constructing the remaining policy functions
- Cost considerations:
  - if $\widetilde{\Psi}^2$ can be computed analytically, then the cost of hybrid is the same than perturbation
  - otherwise, a numeric solver must be used, and the cost can be substantially higher; in this case, from a computational cost point of view, intra-temporal choice conditions should be preferred over inter-temporal conditions for constructing $\widetilde{\Psi}$ (no conditional expectations in the former)

Choosing a hybrid solution (2/2)

Accuracy considerations

- Suppose we have a metric for the distance to the true solution of the perturbation solution:

$$\widehat{\Delta}^i \equiv \left\| \widehat{\Psi}^i \left(\mathbf{x}_t, \mathbf{z}_t \right) - \Psi^i \left(\mathbf{x}_t, \mathbf{z}_t \right) \right\|, \ i = 1, 2$$

- Similarly, assume we have a similar metric for the hybrid solution:

$$\widehat{\Delta}^i \equiv \left\| \widetilde{\Psi}^i \left(\mathbf{x}_t, \mathbf{z}_t \right) - \Psi^i \left(\mathbf{x}_t, \mathbf{z}_t \right) \right\|, \ i = 1, 2$$

- One can show that:
  1. If $\widehat{\Delta}^1 = 0$ and $\widehat{\Delta}^2 > 0$, then any hybrid solution is more accurate than the perturbation solution.
  2. If $\widehat{\Delta}^1 > 0$ and $\widehat{\Delta}^2 = 0$, then any hybrid solution is less accurate than the perturbation solution.

$\Rightarrow$ accuracy of hybrid entirely determined by accuracy of $\widehat{\Psi}^1$

An illustration: one-sector growth model (1/2)

- The model:

$$\max_{\{k_{t+1},c_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s. t. } c_t + k_{t+1} = k_t + a_t f(k_t)$$

$$\ln a_{t+1} = \rho \ln a_t + \varepsilon_{t+1} \qquad \varepsilon_t \sim \mathcal{N}\left(0, \sigma^2\right)$$

- Euler equation:

$$u'(c_t) = \beta E_t \left\{ u'(c_{t+1})\, a_{t+1} f'(k_{t+1}) \right\}$$

- One endogenous state variable k_t, one exogenous state variable a_t and one control variable c_t
- One inter-temporal choice condition (Euler equation, EE) and one intra-temporal choice condition (budget constraint, BC)
- Therefore, four possible hybrid solutions

An illustration: one-sector growth model (2/2)

HYB1: Fix $\widehat{K}(k_t, a_t)$ and define $\widetilde{C}(k_t, a_t) = c_t$ from BC:

$$c_t = k_t + a_t f(k_t) - \widehat{K}(k_t, a_t)$$

HYB2: Fix $\widehat{K}(k_t, a_t)$ and define $\widetilde{C}(k_t, a_t)$ from EE:

$$u'\left(\widetilde{C}(k_t, a_t)\right) = \beta E_t\left\{u'\left[\widetilde{C}\left(\widehat{K}(k_t, a_t), a_{t+1}\right)\right] a_{t+1} f'\left(\widehat{K}(k_t, a_t)\right)\right\}$$

where $a_{t+1} = a_t^{\rho} \exp(\varepsilon_{t+1})$.

HYB3: Fix $\widehat{C}(k_t, a_t)$ and define $\widetilde{K}(k_t, a_t) = k_{t+1}$ from BC:

$$k_{t+1} = k_t + a_t f(k_t) - \widehat{C}(k_t, a_t)$$

HYB4: Fix $\widehat{C}(k_t, a_t)$ and define $\widetilde{K}(k_t, a_t) = k_{t+1}$ from EE:

$$u'\left(\widehat{C}(k_t, a_t)\right) = \beta E_t\left\{u'\left(\widehat{C}(k_{t+1}, a_{t+1})\right) a_{t+1} f'(k_{t+1})\right\}$$

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