Deterministic Models

Perfect foreight, nonlinearities and occasionally binding constraints

Sébastien Villemot

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Introduction

- Deterministic = perfect foresight
- Perfect anticipation of all shocks in the future, and therefore of all future choice variables
- Can be solved exactly (up to rounding errors)
- Full nonlinearities taken into account
- Often useful when starting study of a model, or when studying the effect of strong nonlinearities

Outline

1. Presentation of the problem
2. Solution techniques
3. Shocks: temporary/permanent, unexpected/pre-announced
4. Occasionally binding constraints
5. Extended path
6. Appendix: dealing with nonlinearities using higher order approximation of stochastic models

Outline

The (deterministic) neoclassical growth model

$$\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t.

$$c_t + k_t = A_t k_{t-1}^{\alpha} + (1-\delta) k_{t-1}$$

First order conditions:

$$\begin{aligned} c_t^{-\sigma} &= \beta c_{t+1}^{-\sigma} \left(\alpha A_{t+1} k_t^{\alpha-1} + 1 - \delta \right) \\ c_t + k_t &= A_t k_{t-1}^{\alpha} + (1-\delta) k_{t-1} \end{aligned}$$

Steady state:

$$\begin{aligned} \bar{k} &= \left(\frac{1-\beta(1-\delta)}{\beta\alpha\bar{A}} \right)^{\frac{1}{\alpha-1}} \\ \bar{c} &= \bar{A}\bar{k}^{\alpha} - \delta\bar{k} \end{aligned}$$

Note the absence of stochastic elements! No expectancy term, no probability distribution

The general problem

Deterministic, perfect foresight, case:

$$f(y_{t+1}, y_t, y_{t-1}, u_t) = 0$$

y : vector of endogenous variables

u : vector of exogenous shocks

Identification rule: as many endogenous (y) as equations (f)

Steady state

- A steady state, $\bar{y}$, for the model satisfies

$$f(\bar{y}, \bar{y}, \bar{y}, \bar{u}) = 0$$

- Note that a steady state is conditional to:
    - The steady state values of exogenous variables $\bar{u}$
    - The value of parameters (implicit in the above definition)
- Even for a given set of exogenous and parameter values, some (nonlinear) models have several steady states
- The steady state is computed by Dynare with the `steady` command
- That command internally uses a nonlinear solver

What if more than one lead or one lag?

- A model with more than one lead or lag can be transformed in the form with one lead and one lag using auxiliary variables
- Transformation done automatically by Dynare
- For example, if there is a variable with two leads x_{t+2}:
  - create a new auxiliary variable a
  - replace all occurrences of x_{t+2} by a_{t+1}
  - add a new equation: $a_t = x_{t+1}$
- Symmetric process for variables with more than one lag

Return to the neoclassical growth model

$$y_t = \begin{pmatrix} c_t \\ k_t \end{pmatrix}$$

$$u_t = A_t$$

$$f(y_t) = \begin{pmatrix} c_t^{-\sigma} - \beta c_{t+1}^{-\sigma} \left(\alpha A_{t+1} k_t^{\alpha-1} + 1 - \delta\right) \\ c_t + k_t - A_t k_{t-1}^{\alpha} + (1-\delta) k_{t-1} \end{pmatrix}$$

Solution of deterministic models

- Approximation: impose return to equilibrium in finite time instead of asymptotically
- However possible to return to another point than the steady state
- Useful to study full implications of nonlinearities
- Computes the trajectory of the variables numerically
- Uses a Newton-type method on the stacked system

A two-boundary value problem

Approximation of an infinite horizon model by a finite horizon one

The stacked system for a simulation over T periods:

$$\begin{cases} f(y_2, y_1, y_0, u_1) = 0 \\ f(y_3, y_2, y_1, u_2) = 0 \\ \vdots \\ f(y_{T+1}, y_T, y_{T-1}, u_T) = 0 \end{cases}$$

for y_0 and $y_{T+1} = \bar{y}$ given.

Compact representation:

$$F(Y) = 0$$

where $Y = \begin{bmatrix} y'_1 & y'_2 & \dots & y'_T \end{bmatrix}'$.

Outline

A Newton approach

- Start from an initial guess $Y^{(0)}$
- Iterate. Updated solutions $Y^{(k+1)}$ are obtained by solving:

$$F(Y^{(k)}) + \left[\frac{\partial F}{\partial Y}\right]\left(Y^{(k+1)} - Y^{(k)}\right) = 0$$

- Terminal condition:

$$||Y^{(k+1)} - Y^{(k)}|| < \varepsilon_Y \text{ and/or } ||F(Y^{(k)})|| < \varepsilon_F$$

A practical difficulty

The size of the Jacobian is very large. For a simulation over T periods of a model with n endogenous variables, it is a matrix of order $n \times T$.

3 ways of dealing with it:

- 15 years ago, it was more of a problem than today: LBJ (the default method in Dynare $\leq$ 4.2) exploited the particular structure of this Jacobian using relaxation techniques
- Handle the Jacobian as one large, sparse, matrix (now the default method in Dynare $\geq$ 4.3)
- Block decomposition (divide-and-conquer methods) implemented by Mihoubi

Shape of the Jacobian

$$\frac{\partial F}{\partial Y} = \begin{pmatrix} B_1 & C_1 & & & & & \\ A_2 & B_2 & C_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & A_t & B_t & C_t & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & A_{T-1} & B_{T-1} & C_{T-1} \\ & & & & & A_T & B_T \end{pmatrix}$$

where

$$A_s = \frac{\partial f}{\partial y_{t-1}}(y_{s+1}, y_s, y_{s-1})$$

$$B_s = \frac{\partial f}{\partial y_t}(y_{s+1}, y_s, y_{s-1})$$

$$C_s = \frac{\partial f}{\partial y_{t+1}}(y_{s+1}, y_s, y_{s-1})$$

Relaxation (1/5)

The idea is to triangularize the stacked system:

$$\begin{pmatrix} B_1 & C_1 & & & & \\ A_2 & B_2 & C_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{T-1} & B_{T-1} & C_{T-1} \\ & & & & A_T & B_T \end{pmatrix} \Delta Y = - \begin{pmatrix} f(y_2, y_1, y_0, u_1) \\ f(y_3, y_2, y_1, u_2) \\ \vdots \\ \vdots \\ f(y_T, y_{T-1}, y_T, u_{T-1}) \\ f(y_{T+1}, y_T, y_{T-1}, u_T) \end{pmatrix}$$

Relaxation (2/5)

First period is special:

$$\begin{pmatrix} I & D_1 & & & & \\ & B_2 - A_2 D_1 & C_2 & & & \\ & A_3 & B_3 & C_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{T-1} & B_{T-1} & C_{T-1} \\ & & & & A_T & B_T \end{pmatrix} \Delta Y = - \begin{pmatrix} d_1 \\ f(y_3, y_2, y_1, u_2) + A_2 d_1 \\ f(y_4, y_3, y_2, u_3) \\ \vdots \\ f(y_T, y_{T-1}, y_T, u_{T-1}) \\ f(y_{T+1}, y_T, y_{T-1}, u_T) \end{pmatrix}$$

where

- $D_1 = B_1^{-1} C_1$
- $d_1 = B_1^{-1} f(y_2, y_1, y_0, u_1)$

Relaxation (3/5)

Normal iteration:

$$\begin{pmatrix} I & D_1 & & & & \\ & I & D_2 & & & \\ & & B_3 - A_3 D_2 & C_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{T-1} & B_{T-1} & C_{T-1} \\ & & & & A_T & B_T \end{pmatrix} \Delta Y = - \begin{pmatrix} d_1 \\ d_2 \\ f(y_4, y_3, y_2, u_3) + A_3 d_2 \\ \vdots \\ f(y_T, y_{T-1}, y_T, u_{T-1}) \\ f(y_{T+1}, y_T, y_{T-1}, u_T) \end{pmatrix}$$

where

- $D_2 = (B_2 - A_2 D_1)^{-1} C_2$
- $d_2 = (B_2 - A_2 D_1)^{-1} (f(y_3, y_2, y_1, u_2) + A_2 d_1)$

Relaxation (4/5)

Final iteration:

$$\begin{pmatrix} I & D_1 & & & & \\ & I & D_2 & & & \\ & & I & D_3 & & \\ & & & \ddots & \ddots & \\ & & & & I & D_{T-1} \\ & & & & & I \end{pmatrix} \Delta Y = - \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{T-1} \\ d_T \end{pmatrix}$$

where

$$d_T = (B_T - A_T D_{T-1})^{-1} (f(y_{T+1}, y_T, y_{T-1}, u_T) + A_T d_{T-1})$$

Relaxation (5/5)

- The system is then solved by backward iteration:

$$
\begin{aligned}
y_T^{k+1} &= y_T^k - d_T \\
y_{T-1}^{k+1} &= y_{T-1}^k - d_{T-1} - D_{T-1}(y_T^{k+1} - y_T^k) \\
&\vdots \\
y_1^{k+1} &= y_1^k - d_1 - D_1(y_2^{k+1} - y_2^k)
\end{aligned}
$$

- No need to ever store the whole Jacobian: only the D_s and d_s have to be stored
- Relaxation was the default method in Dynare $\leq$ 4.2, since it was memory efficient

Sparse matrix algebra

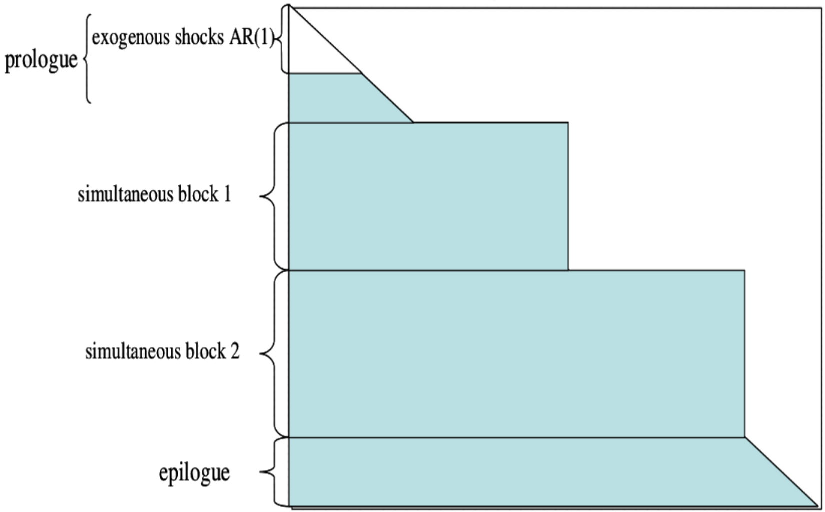
- A sparse matrix is a matrix where most entries are zero
- The Jacobian of the deterministic problem is a sparse matrix:
  - Lots of zero blocks
  - The A_s, B_s and C_s are themselves sparse
- More efficient storage possible than storing all entries
- Usually stored as a list of triplets (i, j, v) where (i, j) is a matrix coordinate and v a non-zero value
- Family of optimized algorithms for such matrices (including matrix inversion for our Newton algorithm)
- Available as native objects in MATLAB/Octave
- Works well for medium size deterministic models
- Nowadays more efficient than relaxation, even though it does not exploit the particular structure of the Jacobian $\Rightarrow$ default method in Dynare $\geq$ 4.3

Block decomposition (1/3)

- Idea: apply a divide-and-conquer technique to model simulation
- Principle: identify recursive and simultaneous blocks in the model structure
- First block (prologue): equations that only involve variables determined by previous equations; example: AR(1) processes
- Last block (epilogue): pure output/reporting equations
- In between: simultaneous blocks, that depend recursively on each other
- The identification of the blocks is performed through a matching between variables and equations (normalization), then a reordering of both

Block decomposition (2/3)

Form of the reordered Jacobian



Block decomposition (3/3)

- Can provide a significant speed-up on large models
- Implemented in Dynare by Ferhat Mihoubi
- Available as option `block` to the `model` command
- Bigger gains when used in conjunction with `bytecode` options

Outline

Example: neoclassical growth model with investment

The social planner problem is as follows:

$$\max_{\{c_{t+j},\ell_{t+j},k_{t+j}\}_{j=0}^{\infty}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, \ell_{t+j})$$

s.t.

$$\begin{aligned} y_t &= c_t + i_t \\ y_t &= A_t f(k_{t-1}, \ell_t) \\ k_t &= i_t + (1-\delta) k_{t-1} \\ A_t &= A^\star e^{a_t} \\ a_t &= \rho\, a_{t-1} + \varepsilon_t \end{aligned}$$

where ε_t is an exogenous shock.

Specifications

- Utility function:

$$u(c_t, \ell_t) = \frac{\left[c_t^{\theta}(1-\ell_t)^{1-\theta}\right]^{1-\tau}}{1-\tau}$$

- Production function:

$$f(k_{t-1}, \ell_t) = \left[\alpha k_{t-1}^{\psi} + (1-\alpha)\ell_t^{\psi}\right]^{\frac{1}{\psi}}$$

First order conditions

- Euler equation:

$$u_c(c_t,\ell_t) = \beta\,\mathbb{E}_t\left[u_c(c_{t+1},\ell_{t+1})\Big(A_{t+1}f_k(k_t,\ell_{t+1})+1-\delta\Big)\right]$$

- Arbitrage between consumption and leisure:

$$\frac{u_\ell(c_t,\ell_t)}{u_c(c_t,\ell_t)} + A_t f_l(k_{t-1},\ell_t) = 0$$

- Resource constraint:

$$c_t + k_t = A_t f(k_{t-1},\ell_t) + (1-\delta)k_{t-1}$$

Calibration

Weight of consumption in utility	θ	0.357
Risk aversion	τ	2.0
Share of capital in production	α	0.45
Elasticity of substitution capital/labor (fct of...)	ψ	-0.1
Discount factor	β	0.99
Depreciation rate	δ	0.02
Autocorrelation of productivity	ρ	0.8
Steady state level of productivity	$A^{\star}$	1

Scenario 1: Return to equilibrium

Return to equilibrium starting from $k_0 = 0.5\bar{k}$.

Fragment from `rbc_det1.mod`

```
...
steady;

ik = varlist_indices('Capital',M_.endo_names);
CapitalSS = oo_.steady_state(ik);

histval;
Capital(0) = CapitalSS/2;
end;

simul(periods=300);
```

Scenario 2: A temporary shock to TFP

- The economy starts from the steady state
- There is an unexpected negative shock at the beginning of period 1: $\varepsilon_1 = -0.1$

Fragment from rbc_det2.mod

```
...
steady;

shocks;
var EfficiencyInnovation;
periods 1;
values -0.1;
end;

simul(periods=100);
```

Scenario 3: Pre-announced favorable shocks in the future

- The economy starts from the steady state
- There is a sequence of positive shocks to A_t: 4% in period 5 and an additional 1% during the 4 following periods

Fragment from rbc_det3.mod

```
...
steady;

shocks;
var EfficiencyInnovation;
periods 4, 5:8;
values 0.04, 0.01;
end;
```

Scenario 4: A permanent shock

- The economy starts from the initial steady state ($a_0 = 0$)
- In period 1, TFP increases by 5% permanently (and this was unexpected)

Fragment from `rbc_det4.mod`

```
...
initval;
EfficiencyInnovation = 0;
end;

steady;

endval;
EfficiencyInnovation = (1-rho)*log(1.05);
end;

steady;
```

Scenario 5: A pre-announced permanent shock

- The economy starts from the initial steady state ($a_0 = 0$)
- In period 6, TFP increases by 5% permanently
- A `shocks` block is used to maintain TFP at its initial level during periods 1–5

Fragment from rbc_det5.mod

```
...
// Same initval and endval blocks as in Scenario 4
...

shocks;
var EfficiencyInnovation;
periods 1:5;
values 0;
end;
```

Outline

Zero nominal interest rate lower bound

- Implemented by writing the law of motion under the following form in Dynare:

$$i_t = \max\left\{0, (1-\rho_i)i^* + \rho_i i_{t-1} + \rho_\pi(\pi_t - \pi^*) + \varepsilon_t^i\right\}$$

- *Warning:* this form will be accepted in a stochastic model, but the constraint will not be enforced in that case!

Irreversible investment

Same model than above, but the social planner is constrained to positive investment paths:

$$\max_{\{c_{t+j},\ell_{t+j},k_{t+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} \beta^j u(c_{t+j},\ell_{t+j})$$

s.t.

$$\begin{aligned} y_t &= c_t + i_t \\ y_t &= A_t f(k_{t-1},\ell_t) \\ k_t &= i_t + (1-\delta)k_{t-1} \\ i_t &\geq 0 \\ A_t &= A^\star e^{a_t} \\ a_t &= \rho\, a_{t-1} + \varepsilon_t \end{aligned}$$

where the technology (f) and the preferences (u) are as above.

First order conditions

$$u_c(c_t,\ell_t) - \mu_t = \beta \mathbb{E}_t\Big[u_c(c_{t+1},\ell_{t+1})\left(A_{t+1}f_k(k_t,\ell_{t+1}) + 1 - \delta\right) \\ - \mu_{t+1}(1-\delta)\Big]$$

$$\frac{u_\ell(c_t,\ell_t)}{u_c(c_t,\ell_t)} + A_t f_l(k_{t-1},\ell_t) = 0$$

$$c_t + k_t = A_t f(k_{t-1},\ell_t) + (1-\delta)k_{t-1}$$

$$\mu_t\left(k_t - (1-\delta)k_{t-1}\right) = 0$$

where $\mu_t \geq 0$ is the Lagrange multiplier associated to the non-negativity constraint for investment.

Writing this model in Dynare

Fragment from `rbcii.mod`

```
mu = max(0,(((c^theta)*((1-l)^(1-theta)))^(1-tau))/c
           - expterm(1)+beta*mu(1)*(1-delta));

(i<=0)*(k - (1-delta)*k(-1))
 + (i>0)*((((c^theta)*((1-l)^(1-theta)))^(1-tau))/c
         - expterm(1)+beta*mu(1)*(1-delta)) = 0;

expterm = beta*((((c^theta)*((1-l)^(1-theta)))^(1-tau))/c)
          *(alpha*((y/k(-1))^(1-psi))+1-delta);
```

Outline

Extended path (EP) algorithm

- Algorithm for creating a stochastic simulated series
- At every period, compute endogenous variables by running a *deterministic* simulation with:
  - the previous period as initial condition
  - the steady state as terminal condition
  - a random shock drawn for the current period
  - but no shock in the future
- Advantages:
  - shocks are unexpected *at every period*
  - nonlinearities fully taken into account
- Inconvenient: solution under certainty equivalence (Jensen inequality is violated)
- Method introduced by Fair and Taylor (1983)
- Implemented in Dynare 4.3 by Stéphane Adjemian under the command `extended_path`

k-step ahead EP

- Accuracy can be improved by computing conditional expectation by quadrature, computing next period endogenous variables with the previous algorithm
- Approximation: at date t, agents assume that there will be no more shocks after period $t+k$ (hence k measures the degree of future uncertainty taken into account)
- If $k=1$: one-step ahead EP; no more certainty equivalence
- By recurrence, one can compute a k-step ahead EP: even more uncertainty taken into account
- Difficulty: computing complexity grows exponentially with k
- k-step ahead EP currently implemented in (forthcoming) Dynare 4.4; triggered with option `order = k` of `extended_path` command

Outline

Local approximation of stochastic models

The general problem:

$$\mathbb{E}_t f(y_{t+1}, y_t, y_{t-1}, u_t) = 0$$

y : vector of endogenous variables

u : vector of exogenous shocks

with:

$$\begin{aligned} \mathbb{E}(u_t) &= 0 \\ \mathbb{E}(u_t u_t') &= \Sigma_u \\ \mathbb{E}(u_t u_s') &= 0 \text{ for } t \neq s \end{aligned}$$

What is a solution to this problem?

- A solution is a policy function of the form:

$$y_t = g\left(y_{t-1}, u_t, \sigma\right)$$

where σ is the *stochastic scale* of the problem and:

$$u_{t+1} = \sigma\,\varepsilon_{t+1}$$

- The policy function must satisfy:

$$\mathbb{E}_t f\left(g\left(g\left(y_{t-1}, u_t, \sigma\right), u_{t+1}, \sigma\right), g\left(y_{t-1}, u_t, \sigma\right), y_{t-1}, u_t\right) = 0$$

Local approximations

$$\hat{g}^{(1)}\left(y_{t+1}, u_t, \sigma\right) = \bar{y} + g_y \hat{y}_{t-1} + g_u u_t$$

$$\begin{aligned}\hat{g}^{(2)}\left(y_{t+1}, u_t, \sigma\right) = {} & \bar{y} + \frac{1}{2} g_{\sigma\sigma} + g_y \hat{y}_{t-1} + g_u u_t \\ & + \frac{1}{2}\left(g_{yy}\left(\hat{y}_{t-1} \otimes \hat{y}_{t-1}\right) + g_{uu}\left(u_t \otimes u_t\right)\right) \\ & + g_{yu}\left(\hat{y}_{t-1} \otimes u_t\right)\end{aligned}$$

$$\begin{aligned}\hat{g}^{(3)}\left(y_{t+1}, u_t, \sigma\right) = {} & \bar{y} + \frac{1}{2} g_{\sigma\sigma} + \frac{1}{6} g_{\sigma\sigma\sigma} + \frac{1}{2} g_{\sigma\sigma y} \hat{y}_{t-1} + \frac{1}{2} g_{\sigma\sigma u} u_t \\ & + g_y \hat{y}_{t-1} + g_u u_t + \ldots\end{aligned}$$

Breaking certainty equivalence (1/2)

The combination of future uncertainty (future shocks) and nonlinear relationships makes for precautionary motives or risk premia.

- 1st order: certainty equivalence; today's decisions don't depend on future uncertainty
- 2nd order:

$$\begin{aligned}\hat{g}^{(2)}\left(y_{t+1}, u_t, \sigma\right) = {} & \bar{y} + \frac{1}{2} g_{\sigma\sigma} + g_y \hat{y}_{t-1} + g_u u_t \\ & + \frac{1}{2}\left(g_{yy}\left(\hat{y}_{t-1} \otimes \hat{y}_{t-1}\right) + g_{uu}\left(u_t \otimes u_t\right)\right) \\ & + g_{yu}\left(\hat{y}_{t-1} \otimes u_t\right)\end{aligned}$$

Risk premium is a constant: $\frac{1}{2} g_{\sigma\sigma}$

Breaking certainty equivalence (2/2)

- 3rd order:

$$\hat{g}^{(3)}\left(y_{t+1}, u_t, \sigma\right) = \bar{y} + \frac{1}{2} g_{\sigma\sigma} + \frac{1}{6} g_{\sigma\sigma\sigma} + \frac{1}{2} g_{\sigma\sigma y} \hat{y}_{t-1} + \frac{1}{2} g_{\sigma\sigma u} u_t + g_y \hat{y}_{t-1} + g_u u_t + \ldots$$

Risk premium is linear in the state variables:

$$\frac{1}{2} g_{\sigma\sigma} + \frac{1}{6} g_{\sigma\sigma\sigma} + \frac{1}{2} g_{\sigma\sigma y} \hat{y}_{t-1} + \frac{1}{2} g_{\sigma\sigma u} u_t$$

The cost of local approximations

1. High order approximations are accurate around the steady state, and more so than lower order approximations
2. But can be totally wrong far from the steady state (and may be more so than lower order approximations)
3. Error of approximation of a solution $\hat{g}$, at a given point of the state space (y_{t-1}, u_t):

$$\mathcal{E}\left(y_{t-1}, u_t\right) = \\ \mathbb{E}_t f\left(\hat{g}\left(\hat{g}\left(y_{t-1}, u_t, \sigma\right), u_{t+1}, \sigma\right), \hat{g}\left(y_{t-1}, u_t, \sigma\right), y_{t-1}, u_t\right)$$

4. Necessity for pruning

Approximation of occasionally binding constraints with penalty functions

The investment positivity constraint is translated into a penalty on the welfare:

$$\max_{\{c_{t+j},\ell_{t+j},k_{t+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} \beta^j u(c_{t+j},\ell_{t+j}) + h \cdot \log(i_{t+j})$$

s.t.

$$\begin{aligned}
y_t &= c_t + i_t \\
y_t &= A_t f(k_{t-1},\ell_t) \\
k_t &= i_t + (1-\delta)k_{t-1} \\
A_t &= A^\star e^{a_t} \\
a_t &= \rho\, a_{t-1} + \varepsilon_t
\end{aligned}$$

where the technology (f) and the preferences (u) are as before, and h governs the strength of the penalty (*barrier parameter*)