The Sources of Sovereign Risk
A calibration based on Lévy stochastic processes☆

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Abstract

Governments choose to issue risky or riskless debt depending on the nature of the stochastic process of output. We use Brownian motion and Poisson shocks—a modeling method in the literature on corporate default known as Lévy processes—to approximate a decomposition of the output process into a smooth and a jump component. Using an Eaton and Gersovitz (1981) model of debt repudiation, we show that the Brownian part explains the counter-cyclical behavior of the current account, and the Poisson part explains the risk of default—thus enabling our model to account for key stylized facts regarding sovereign risk.

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1. Introduction

The literature on sovereign debt has attempted to fit simultaneously the data on debt levels and on default probabilities. In the models inspired by Eaton and Gersovitz (1981), the modeling trade-off has been the following. Either the cost of default is set to low levels: in that case the country will often default, as in the data, but will also be rationed to abnormally low levels of debt compared to historical averages. Or the other way around: if the cost of default is high, then the country can borrow in quantities observed in the data, but it will then default too rarely compared to real life. Original work by Aguiar and Gopinath (2006) or Arellano (2008) for instance successfully matched the probability of default but at the expense of very low sustainable debt levels. A newer generation of models (notably Hatchondo and Martinez, 2009, followed by Chatterjee and Eyigungor, 2012 and Bianchi et al., 2013) has managed to fit both moments by postulating a pro-cyclical cost of default, as in Arellano (2008). In these models, a recession has the potential to create a default, as the cost becomes virtually nil when the output gap is large, while very expensive in good

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states. Further insights have been gained in these papers by incorporating a richer structure regarding the maturity of debt. Long-term debt, by reducing the refinancing risk, allows countries to borrow more than suggested by the short-term debt model.

Another major fact emphasized by this literature has to do with the counter-cyclicality of the current account. When output is rising, countries typically borrow more, and the opposite happens when it is falling. This fact is at odds with the permanent income version of the first generation of balance of payments models.

In the wake of these researches, our paper provides a new perspective for characterizing sovereign debt profiles. Our key insight is that the pattern of output (*i.e.* the stochastic structure of GDP) is critical in determining a country's borrowing and default decisions. This idea—that the nature of uncertainty about the fundamental matters—is standard in structural models of firm default. By drawing on the mathematical theory of Lévy processes, this strand of the corporate finance literature has successfully addressed several puzzles regarding default risk and credit spreads (from the early works referenced in Sundaresan (2000) to the present day).

Our paper builds on this approach and shows how incorporating both a smooth[1] ("Brownian") and a jump ("Poisson") component into the GDP process helps shedding light on the issues and challenges to the sovereign debt literature mentioned above. Specifically, our main results are the following.

In the context of sovereign risk, in which default is an option to be compared to the alternative of continuing to service the debt, we reach the striking conclusion that when debt is short-term, default is only driven by the Poisson part of the GDP process. A smooth stochastic process does not produce default for reasons similar to the deterministic case: with the arrival of new information, the sovereign always prefers to adapt its debt rather than risking to drift towards a costly default. On the other hand, the smooth part of the stochastic process produces a contra-cyclical current account, consistent with empirical evidence. This feature is not captured by the Poisson shocks. In other words, two key stylized facts regarding debtor countries, counter-cyclicality of their current account and high exposure to risk of default, while obviously stemming from a common cause—the willingness to borrow extensively—do not proceed from the same risk factor. Our analysis indicates that the data of a debt-to-GDP ratio is not by itself sufficient to determine whether a sovereign will default: the nature of shocks to GDP is key to understand a country's level and terms of borrowing and default decisions.

How do our results compare to standard models of corporate default? In a Merton model of default (Merton, 1974), a Brownian process does generate a risk of default at maturity. The key reason behind the discrepancy between the Merton models and ours has to do not only with the endogeneity of default, but with the endogeneity and the frequency of the decision to borrow. The borrower, in our model, has the ability to control *frequently* its debt, adapting to the fundamental. In the original Merton model, debt is exogenous and default

[1] Throughout the paper, "smooth" refers to a continuous process or an approximation of such a process, not to stronger notions of regularity.

can only occur at maturity. In other seminal models of corporate finance (Leland, 1994 and Leland and Toft, 1996), default is endogenous; in these models, however, the decision to borrow is only taken once. Obviously, if the frequency of decision-making is too small to enable the country to adjust to shocks to the fundamental, default can't be ruled out, whatever the shape of the fundamental process. Indeed, in our own framework, a longer maturity has the potential of raising the risk of default (as in Nuño and Thomas, 2015 or Hatchondo et al., 2016).

The paper proceeds as follows. Section 2 presents a review of the existing models on sovereign debt. Section 3 presents the model setup. Section 4 analyzes how default in equilibrium depends on the nature (Brownian or Poisson) of the stochastic process driving output. We use the discretized version on the Brownian model developed by Cox et al. (1979) to show the key result of the paper: at equilibrium, the risk of default arises from the Poisson stochastic process only, not from the Brownian part. Section 5 presents calibrated results illustrating the properties of the theoretical model; we also analyze how the maturity structure of debt comes into the picture. We show that the Brownian part is key to understand the countercyclical nature of the current account. In line with the results obtained by Nuño and Thomas (2015) and Hatchondo and Martinez (2009), we also show that long maturities are associated with positive default, although, in the context of our model, we also show that such strategy is not welfare improving. Section 6 offers a deeper look at the properties of the model in the limit of continuous time: in this limit, we show that the country must entirely renounce leveraging the Brownian part of GDP; by constrast, it does leverage the Poisson part. Section 7 concludes.

2. Calibrating sovereign debt models

Calibrated models of sovereign debt owe much to the papers by Aguiar and Gopinath (2006), Arellano (2008) and Mendoza and Yue (2012), which followed in the earlier tradition of Eaton and Gersovitz (1981), Cohen and Sachs (1986) and Bulow and Rogoff (1989).

Their framework is as follows. An indebted country can decide to default, paying as a consequence a lump-sum cost. This cost sets the upper limit of debt. These models have successfully reproduced key business cycle correlations regarding aggregate spending and balance of payments in particular. Table 1 summarizes the key results obtained by several recent papers along in terms of debt-to-GDP ratio and default probabilities.

Before discussing the results of these papers, one should note the improbably high discount factor that some models have to rely on to sustain their equilibrium. For example, Yue (2010) and Aguiar and Gopinath (2006) set respective values of 0.72 and 0.8 for the (quarterly!) discount factor. This high impatience helps to generate frequent defaults and a desire to hold debt, but it is unrealistically high, even when accounting for political instability. Others, like Arellano (2008), Benjamin and Wright (2009) or Chatterjee and Eyigungor (2012) use values close to 0.95, which is more realistic.

In order to fit the conventional wisdom of markets and international financial institutions, one would want a model that could predict:

Table 1: Overview of mean debt-to-GDP ratios and default probabilities in the literature

Paper	Main feature	Debt-to-GDP mean ratio (%, annual)	Default probability (%, annual)
Arellano (2008)	Non-linear default cost	1	3.0
Aguiar and Gopinath (2006)	Shocks to GDP trend	5	0.9
Cuadra and Sapriza (2008)	Political uncertainty	2	4.8
Fink and Scholl (2011)	Bailouts and conditionality	1	5.0
Yue (2010)	Endogenous recovery	3	2.7
Hatchondo and Martinez (2009)	Long-duration bonds	5	2.9
Benjamin and Wright (2009)	Endogenous recovery	16	4.4
Chatterjee and Eyigungor (2012)	Long-duration bonds	18	6.8
Mendoza and Yue (2012)	Endogenous default cost	23	2.8
Bianchi et al. (2013)	Gross financial assets	42	—
Hatchondo et al. (2014)	Non-defaultable debt	66	—

Most papers report the debt-to-GDP ratio using GDP measured at a quarterly frequency; we choose to use GDP measured at an annual frequency, since this is the convention used by policymakers and in the policy debate. For Aguiar and Gopinath (2006), we report results for their model II (with shocks to GDP trend). For Arellano (2008) and Aguiar and Gopinath (2006), the reported values come from Hatchondo et al. (2010) who resimulate these models using more precise numerical techniques. For Hatchondo and Martinez (2009), the reported values are those obtained for their λ parameter equal to 20%. For Bianchi et al. (2013) and Hatchondo et al. (2014), the default probabilities are not reported in the papers (they only report spreads on long-term bonds).

- *Threshold debt levels in the vicinity of 40% of yearly GDP.* The mean debt-to-GDP ratio in 2009 was 42% across countries, according to our computations using World Bank (2010). Note that World Bank (2004) classifies as "severely indebted" countries with a debt-to-GNI ratio above 80%, and as "moderately indebted" countries with a ratio above 48%: our target of 40% is therefore in the lower end of the range of interest.

- *Annual default probabilities in the range of 4 to 5%.* Yue (2010) reports that the average default rate of Argentina since 1824 is 2.7%. Benjamin and Wright (2009) estimate an average default rate across countries of 4.4% for the period 1989–2006. In the data collected by Cohen and Valadier (2011) over the period 1970–2007, which includes "soft defaults" such as IMF loans, an even higher probability of default of about 7% is documented.

Even though many papers listed in Table 1 reach the target in terms of default probabilities, most fail with respect to the sustainable debt ratios. Only the last two ones (Bianchi et al., 2013 and Hatchondo et al., 2014)[2] are able to replicate realistic levels of debt-to-GDP ratios. The success of those two papers in this dimension seems to be related to the specific form of the cost of default that they assume, which is both high on average and highly pro-cyclical. In Bianchi et al. (2013) (resp. Hatchondo et al., 2014), the cost of default is 17% (resp. 35%) of GDP when the latter is at its steady-state value, which seems rather high. And in Hatchondo et al. (2014), the cost of default is approximately 23% (resp. 46%) of GDP when the latter is 5% below (resp. above) its trend, which corresponds to a very high pro-cyclicality of the default cost.

Although the introduction of a highly pro-cyclical default cost was a critical step forward in making the models quantitatively relevant, this modeling is somewhat at odds with the results in Tomz and Wright (2007), for instance, which shows that over the very long run, the relationship between default and recession is not automatic as recessions, in particular, do not necessarily cause a default. In our own modeling strategy, recessions may or not create a default, depending on the way they are engineered.

3. The setup

3.1. The output process

Models of corporate default have largely benefited from the integration of Lévy processes into the modeling of the fundamental, which is typically the return on firm value. Lévy processes are continuous-time processes with independent and stationary increments satisfying a regularity condition. They generalize the Brownian motion when one removes the

[2]Note that Benjamin and Wright (2009) argue that the historical average of the yearly debt-to-GDP ratio over their data set is precisely 18%. They choose their calibration in order to match that target and are able to do so (they simulate an average 16% debt-to-GDP) using a relatively low value for the output cost of default. Their model may therefore be able to reach higher levels of debt while still keeping the output cost at a reasonable value, but we did not check that.

continuity assumption. The Lévy-Itô decomposition states that any Lévy process is the sum of three terms: a Brownian process with deterministic drift, a compound Poisson process, and a third term which intuitively represents an infinite sum of infinitesimally small jumps.[3]

In models of corporate default, the fundamental is the firm value—an expected discounted sum of cash flows—and the return on firm value is assumed to be Lévy. Similarly, our fundamental can be seen as the sovereign's wealth, an expected discounted sum of output flows, with a return on wealth which is Lévy. Specifically, we assume that output Y_t is subject to i.i.d. growth shocks, *i.e.* that $\log Y_t$ is a Lévy process.[4]

We consider a discrete-time economy where periods are of length h and we will let the time horizon h shrink towards small values.[5] In discrete time, Y_t induces a Markov chain $(Y_{kh})_{k\geq 0}$. We shall compare two sequences of Markov chains, the logarithm of which approaches respectively a Brownian motion and a Poisson process. The specific functional forms used are spelled out in section 4, where we characterize the equilibrium.

3.2. Financial markets

3.2.1. Financial environment

The world financial markets are characterized by an instantaneous, constant, riskless interest rate r. Lenders are risk-neutral and subject to a zero-profit condition by competition. We further suppose that all debt is short-term and needs to be refinanced every period (this assumption will be relaxed later).

The timing of events is as follows. First assume that the country has incurred a debt obligation D_t due at time t, and has always serviced it in full in previous years. At the beginning of period t, the country learns the value of its output Y_t. It then either defaults on its debt or reimburses it. If the debt is reimbursed in full, the country can contract a new loan, borrowing L_t, which must be repaid at time $t+h$ in the amount of D_{t+h}. Note that the implicit instantaneous interest rate is equal to $\frac{\log(D_{t+h}/L_t)}{h}$. Such financial agreements being concluded, if the country services its debt in full it eventually consumes:

$$C_t = Y_t + L_t - D_t.$$

Alternatively, in the event of a debt crisis the country may default (see below). This occurs when output is too low to allow the country to service its debt. Call $\pi_{t+h|t}$ the probability of default at time $t+h$ from the perspective of date t. Then the zero-profit condition for creditors may be written as:

$$L_t e^{rh} = D_{t+h}(1 - \pi_{t+h|t}). \tag{1}$$

Note that we have assumed that investors recover nothing in case of default. This assumption is lifted below in section 5.2.

[3] The third term is a mathematical technicality which we will not need in the present paper.

[4] Notice that expected future output (and therefore wealth) are proportional to current output: $\mathbb{E}_t[Y_{t+kh}] = \mathbb{E}_t\left[\frac{Y_{t+kh}}{Y_{t+(k-1)h}} \dots \frac{Y_{t+h}}{Y_t}\right] Y_t = g_0^k Y_t$ where $g_0 = \mathbb{E}\left[\frac{Y_h}{Y_0}\right]$ since growth shocks are i.i.d..

[5] For now, the discretization step h, which corresponds to (inverse of) the frequency of the decision making, is equal to the maturity of debt. In Section 5.3, we show that this assumption can be relaxed (*i.e.* we can fix the maturity of debt independently of h) with similar results, as long as maturity is not too large.

3.2.2. Default

At any time t, a country that has accumulated a debt D_t may decide to default. When it does so, we assume that the country suffers a penalty $\lambda \in [0, 1)$ on output as a consequence of the crisis. This penalty is captured by no one and is therefore a net social loss. We call Y_t^d the post-penalty value of income (which we distinguish from output) and for the time being simply write:

$$Y_t^d = (1 - \lambda)Y_t.$$

As another cost, we assume that the country is subject to financial autarky, being unable to borrow again later on. We then write consumption as:

$$C_t^d = Y_t^d = (1 - \lambda)Y_t.$$

A milder form of a sanction would be, more realistically, that the country is barred from the financial market for some time only, as in Aguiar and Gopinath (2006). We explore this less demanding assumption in section 5.2. Note however that in the case where the creditors can impose a take it or leave it post-default pattern, the recovery rate is set at the maximum level feasible which in practice is tantamount to imposing financial autarky forever after a default (see Cohen and Villemot, 2015, for a further analysis, with a discussion on the possibility of self-fulfilling debt crises that such models can generate).

3.3. Preferences and equilibrium

3.3.1. Preferences

The decision to default or to stay in financial markets involves a comparison of two paths that imply expectations over the entire future. In order to address this problem, we assume that the country seeks to solve:

$$J^*(D_t, Y_t) = \max_{\{C_{t+kh}\}_{k\geq 0}} \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} e^{-\rho kh} u(C_{t+kh}) \right\},$$

where ρ is the instantaneous rate of preference for the present. D_t can be negative if the country builds up foreign assets reserves. We assume utility is isoelastic of the form:

$$u(C_t) = \frac{C_t^{1-\gamma}}{1 - \gamma}.$$

We call:

$$J^d(Y_t) = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} e^{-\rho kh} u(Y_{t+kh}^d) \right\}$$

the post-default level of utility, which is by construction independent of debt, and to which the country is nailed down in case of servicing difficulties. If it were to stay current on its debt obligation, the country would obtain:

$$J^r(D_t, Y_t) = \max_{L_t, D_{t+h}} \left\{ u\left(Y_t + L_t - D_t\right) + e^{-\rho h} \mathbb{E}_t[J^*(D_{t+h}, Y_{t+h})] \right\},$$

subject to the zero-profit condition (1).

When choosing between how much it can get by staying in the markets and the post-default level of welfare, the country chooses its optimum level:

$$J^*(D_t, Y_t) = \max\{J^r(D_t, Y_t), J^d(Y_t)\}.$$

3.3.2. Recursive equilibrium

We define a recursive equilibrium in which the government does not have commitment and the various agents act sequentially.

The aggregate state of the model is $s = (\delta, D, Y)$, where δ is past credit history (equal to 1 if a country is barred from financial markets, 0 otherwise), D is the stock of debt due in the current period (necessarily equal to zero if $\delta = 1$) and Y is current GDP.

Definition 1 (Recursive equilibrium)**.** *The recursive equilibrium for this economy is defined as a set of policy functions for (i) the government's default decision $\tilde{\delta}'(s)$; (ii) the government's decision for tomorrow's debt holding $\tilde{D}'(s)$; and (iii) the investor's supply of borrowing $\tilde{L}(s, D')$ such that:*

- *given the investor's policy function, $\tilde{\delta}'(s)$ and $\tilde{D}'(s)$ satisfy the government optimization problem:*

$$\tilde{\delta}'(s) = \begin{cases} 1 & \text{if } \delta = 1 \text{ (default in the past) or } J^d(Y) > J^r(D, Y) \text{ (default now)} \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{D}'(s) = \begin{cases} \arg\max_{D'} \left\{ u(Y - D + \tilde{L}(s, D')) + e^{-\rho h}\mathbb{E}_Y[J^*(D', Y')] \right\} & \text{if } \tilde{\delta}'(s) = 0 \\ 0 & \text{otherwise} \end{cases}$$

where:

$$\begin{aligned} J^r(D, Y) &= \max_{D'} \left\{ u(Y - D + \tilde{L}(s, D')) + e^{-\rho h}\mathbb{E}_Y[J^*(D', Y')] \right\} \\ J^d(Y) &= u((1 - \lambda)Y) + e^{-\rho h}\mathbb{E}_Y J^d(Y') \\ J^*(D, Y) &= \max\{J^r(D, Y), J^d(Y)\}; \end{aligned}$$

- *given the government's default decision function, $\tilde{L}(s, D')$ satisfies the zero-profit constraint:*

$$\tilde{L}(s, D') = e^{-rh} \left[1 - \mathbb{E}_Y \tilde{\delta}'(\delta, D', Y') \right] D'.$$

4. Equilibrium under Brownian or Poisson shocks

We now characterize the equilibrium by distinguishing the two polar instances of Lévy processes: the Brownian case and the Poisson case. We use approximations of these processes in order to be able to work in discrete time. This section presents and discusses analytical results. In appendix Appendix B we report the equilibrium policy, price and value functions for the two models studied here.

4.1. The Cox-Ross-Rubinstein case

To approximate the Brownian process, in this section we use the Cox, Ross, and Rubinstein (1979) (henceforth CRR) model, which has been extensively used in the option pricing literature. It provides us with a simple way to approximate Brownian shocks. Here, the law of motion of the log output is a discrete-time version of a Brownian process:

$$Y_{t+h} = \begin{cases} e^{\sigma\sqrt{h}} Y_t & \text{with probability } \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{h} \\ e^{-\sigma\sqrt{h}} Y_t & \text{with probability } \frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{h}. \end{cases}$$

As h goes to zero, this process converges towards a geometric Brownian process of "percentage drift" μ and "percentage volatility" σ. The result that we obtain is quite remarkable:

Proposition 2. *Let $\kappa = \frac{\mu}{\sigma}$. In the CRR model, if $h < h^* = \frac{1}{(\kappa+4\sigma)^2}$, only two cases are possible (for a given initial value of the debt-to-GDP ratio):*

- *the country immediately defaults;*
- *the country never defaults (whatever the future path of output).*

Technically, if the aggregate state of the model is initially $s_0 = (0, D_0, Y_0)$, then either $\tilde{\delta}'(s_0) = 1$, or $\tilde{\delta}'(s_k) = 0$ for all future realizations of s_k.

Proof. See appendix Appendix A. □

In other words, either the debt is already too high and the country immediately defaults, or it will never do so. Starting, say, with zero debt, a country will then never default!

In the CRR model with small time windows, the country has the possibility, if it wants, to monitor its debt safely; it suffices for consumption to follow exactly the ups and downs of the economy. In practice, how small is small? One can compute the length of the critical time window h^* derived in Proposition 2. For a reasonable parameterization of the model, namely with $\kappa = \mu/\sigma$ near one, the time window is about one quarter. Note, moreover, that this is a lower bound: Proposition 2 does not state that the CRR model necessarily features default for any $h > h^*$.

It is worth adding some comments about the interpretation of Proposition 2. First, note that it is relevant to express the bound h^* in terms of the parameters $\kappa = \frac{\mu}{\sigma}$ and σ (instead of using μ and σ). Indeed, κ commands the probability of an up (resp. down) move of the GDP, $\frac{1}{2} + \frac{\kappa\sqrt{h}}{2}$ (resp. $\frac{1}{2} - \frac{\kappa\sqrt{h}}{2}$). Fixing κ, an increase in σ does not change the probability of an up or down move, just the spread between the possible next values of GDP. We see that an increase in this spread has the effect of decreasing h^*. This is consistent with our insight that a short maturity allows the country to adjust to shocks as they arrive. When risk is higher, shorter maturities are needed to monitor consumption and avoid default.

4.2. The Poisson case

The other polar case of the Lévy decomposition is when the law of motion of the log of output corresponds to a discrete-time version of a compound Poisson process:

$$Y_{t+h} = \begin{cases} e^{gh} Y_t & \text{with probability } e^{-p_0 h} \\ e^{gh} k\, \tilde{m}_t\, Y_t & \text{with probability } 1 - e^{-p_0 h}, \end{cases}$$

where $(\tilde{m}_{kh})_{k \geq 0}$ is i.i.d. and $g \geq 0$. For the purpose of our economic analysis, we shall assume that the support of $\tilde{m}$ is included in the interval $(0, 1)$, and therefore represents a "malus" (*i.e.* with a very small probability, the country loses a significant amount of output). The term $k = \frac{p_0 h}{1 - e^{-p_0 h}}$ is a technical artifact of the discretization,[6] and it goes to 1 as h goes to 0.

As h goes to zero, $\log Y_t$ converges towards a compound Poisson process whose rate is p_0, whose jump size distribution equals the distribution of $\log \tilde{m}_t$ and with deterministic trend g.

We arrive at the following result:

Proposition 3. *In the Poisson case, the probability of default (starting from a non-default state at time t) between dates t and $t+1$ is below or equal to $1 - e^{-p_0}$ (independently of the value of h):*

$$\forall s_t = (0, D_t, Y_t) \mid \tilde{\delta}'(s_t) = 0,\ \mathbb{P}\left(\tilde{\delta}'(s_{t+1}) = 1\right) \leq 1 - e^{-p_0}$$

Moreover, this upper bound is attained for some specific parameterizations of the model.

Proof. See appendix Appendix A. □

The comparison between the CRR and Poisson cases is straightforward. When the economy is smooth, countries can continuously adjust their debt levels and never default. When the economy is disrupted by a (large) Poisson shock, default becomes a possibility (of probability p_0 per unit of time, using a first order approximation).

4.3. Intuition

In order to grasp the intuition behind this result, consider a simple 2 periods version of the model. In period 1, output is Y_1, consumption is C_1 and the country borrows $L_1 = C_1 - Y_1$. In period 2, output Y_2 can take two values: Y_2^+ with probability $1-p$ and Y_2^- with probability p (with $Y_2^+ > Y_2^-$).

[6] To be precise, k corresponds to the expectation of the number of shocks of the continuous Poisson process during a period of length $p_0 h$, conditional on the fact that there is at least one shock in this interval.

In this simple setting, the cost of default boils down to losing λY_2 in period 2. If the country were to ask for a riskless loan, it could borrow $L_1 = \frac{\lambda Y_2^-}{1+r}$. If it were to borrow on risky terms, then it would get at most $L_1 = \frac{\lambda Y_2^+(1-p)}{1+r}$.

The comparison of the two lines of credit boils down to a comparison between p and $\omega = \frac{Y_2^+ - Y_2^-}{Y_2^+}$. When the magnitude of the recession is low compared to its risk, *i.e.* $p > \omega$, the country should stay safe. There is no reason to risk a default, for a small benefit. This is typically the case of CRR, in which p is about one half, and the growth rates in good and bad states have a small discrepancy. This is just the opposite of the Poisson case, in which p is low, but ω is high.

When the probability of a bad event is large, while the bad event is itself not that bad, it is not worthy to run the risk of a default. When, conversely, the bad event is quite bad, but the probability that it may occur low, then the borrower will take the risk of a default should the bad event occurs.

We can also understand why neither the discount factor nor the functional form of the utility function enter Proposition 2. Opting for risky debt has an adverse pricing impact which forces the country to consume less than what it would get under the safe strategy. Therefore, taking on default risk when $h < h^*$ is a dominated choice, regardless of the preference parameters. For the same reason, the magnitude of the sanction upon default does not impact the value of h^*.

5. Simulations

5.1. Discretization step and default probability

We now simulate models based on the results derived above. We verify that defaults are generated by the Poisson case only, as the discretization step goes down. Our strategy is to compare two Markov chains of output (CRR and Poisson). We set up a benchmark where the approximations given by the CRR and the Poisson cases described above coincide for some time window h_0 (we pick $h_0 = 4$, *i.e.* one year) and analyze how the properties of the two models diverge for smaller time windows, $h < h_0$. As h becomes smaller, the two Markov chains of output we compare stand in sharper contrast, as one converges to a Brownian motion, while the other one approaches a Poisson process. In order to make a relevant comparison, we impose the following conditions:

i) for all values of h, the two processes have the same average growth;
ii) the magnitude of the output loss in case of a bad Poisson shock does not depend on h;
iii) the approximations coincide when the time window is h_0 (*i.e.* same probabilities for up and down moves, same jump sizes at the yearly level), as mentioned above.

For the CRR case (section 4.1), we set a benchmark with $\sigma = 2.2\%$ and $\mu = 1\%$. The other parameters regarding preferences and the cost of default are presented in Table 2. Except for GDP, the parameters are borrowed from Arellano (2008) and Aguiar and Gopinath (2006).

Table 2: Calibration of discretized Lévy models

Risk aversion	γ	2
Discount rate	ρ	log(0.8)
Riskless interest rate	r	log(1.01)
Loss of output in autarky (% of GDP)	λ	0.5%
Drift of Brownian process	μ	1%
Volatility of Brownian process	σ	2.2%
Period size for which Brownian and Poisson are observationally equivalent	h_0	4

Quarterly frequency.

For the Poisson case (section 4.2), we let

$$Y_{t+h} = \begin{cases} g^+(h)Y_t & \text{with probability } e^{-p_0 h} \\ g^-(h)Y_t & \text{with probability } 1 - e^{-p_0 h}, \end{cases}$$

where g^+ and g^- are functions of h which we define below so as to satisfy conditions i), ii) and iii) above. These conditions imply the following relationships:

$$\begin{aligned} e^{-p_0 h_0} &= \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{h_0} \\ \forall h,\ e^{-p_0 h} g^+(h) + (1 - e^{-p_0 h}) g^-(h) &= \left(\frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{h}\right) e^{\sigma\sqrt{h}} + \left(\frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{h}\right) e^{-\sigma\sqrt{h}} \\ \forall h,\ g^-(h)\frac{p_0 h}{1 - e^{-p_0 h}} &= e^{-\sigma\sqrt{h_0}} \frac{p_0 h_0}{1 - e^{-p_0 h_0}}, \end{aligned}$$

which identify p_0, $g^+(h)$ and $g^-(h)$.

This choice of the Poisson approximation is a particular instance of the specification of section 4.2: take $e^{gh} = g^+(h)$ and $k\tilde{m}_t = e^{-gh} g^-(h)$.

We run the discrete-time approximation of the two models with different values of h ranging from 4 (yearly approximation) to 0.33 (monthly discretization) and 0.08 (weekly discretization). The corresponding threshold for h under which defaults are impossible in this model, as given by proposition 2, is $h^* \simeq 3.4$. This implies, for instance, that at the quarterly decision level, $h = 1$, default would be impossible.

Table 3 reports the results. Note that, by construction, the results are the same for the two processes at the yearly frequency ($h = h_0 = 4$).

One can see that Table 3 illustrates proposition 2: for $h < h^*$, the CRR model has zero default. Conversely, the Poisson case still exhibits defaults as h becomes small (up to one week).

One also sees that the equilibrium debt-to-GDP ratio rises as h falls down to the monthly rate, and then falls as the time horizon becomes weekly. The reason is explained in the last section of the paper.

Table 3: Moments of interest for discretized Lévy processes and various period durations

Period duration (h, in quarters)	4	2	1	0.33	0.08
Discretized Brownian process					
Default threshold (debt-to-GDP, quarterly, %)	48.6	51.9	68.7	82.6	20.5
Default probability in 10 years (%)	30.9	0.0	0.0	0.0	0.0
Discretized Poisson process					
Default threshold (debt-to-GDP, quarterly, %)	48.6	47.9	47.6	47.3	50.8
Default probability in 10 years (%)	30.8	32.9	35.1	35.3	35.1

The processes are parameterized as described in section 5.1. The solution to the detrended model is approximated using value function iteration on a grid of 40 points for the debt-to-GDP ratio. Moments are obtained by averaging over 5,000 simulations of a length of 10 years, starting with no debt.

5.2. Business cycle properties

In this section, we want to test the power of each assumption (Brownian versus Poisson) on another dimension that has been the focus of recent work on sovereign debt models, namely the covariation of the current account with the business cycle.

As emphasized by Arellano (2008) and others, the current account of a typical emerging country is negatively correlated to the business cycle, in contrast to the standard prediction of a permanent income approach to aggregate spending (by which the country should save when output is high).

In order to study second order moments of the model, we need to introduce some ergodicity. In other words, it is necessary to introduce a possibility of moving back from the default state to the normal state. Consequently, and following Arellano (2008), we assume in this section that a defaulting country can return to financial markets after a while. We call x the probability of a settlement at a given period. When the settlement occurs, the penalty λ is lifted. Once a settlement is reached, debt is not canceled; it is only written down to a level consistent with the post-default level of output and the historical data on post-default haircuts. We call V the settlement value of post-default debt.

Indeed, as documented by Sturzenegger and Zettelmeyer (2007) and Cruces and Trebesch (2013), creditors do capture a recovery value of debt after default. Cohen (1992) also showed that post-Brady recovery values were quite significant in the eighties. Even in the most celebrated default incident, Argentina, creditors clawed back about one third of their claims. By taking into account the post-default recovery value, the model can sustain significantly higher levels of debt.

In papers such as Yue (2010) and Benjamin and Wright (2009), the pair (x, V) has been modeled as the endogenous outcome of a bargaining process following a default. Contrary to those authors, but similarly to Hatchondo et al. (2014), we assume that the recovery value is not a function of past debt. The idea is simply that, after a default, prior commitments become irrelevant. In a Nash bargaining framework, creditors and debtors would split among themselves the benefit of avoiding default, which is essentially a function of lost output.

Also note that if the recovery value V were high enough (for example greater than the debt threshold above which the country defaults), then the resulting model would be conceptually equivalent to a model where no settlement is ever reached after a default (*i.e.* where $x = 0$).[7]

For the calibration used in this section, we pick the value $x = 0.05$ (equivalent to an exclusion of five years from financial markets). This is twice the value chosen by Arellano (2008). Our assumptions are in line with the work of Cohen and Valadier (2011) or Gelos et al. (2011), but the specific numbers are not critical to the analysis. We also set $\lambda = 1\%$, which is two times the value used in the previous section: because of the possibility of returning to the markets, the cost of a default is now lower *ceteris paribus*, so we have to raise λ in order to maintain a debt-to-GDP ratio similar to that of the previous section. For the same reason, we set the recovery value V to 20% of quarterly GDP.

The GDP process is the same in both cases (CRR and Poisson) as in section 5.1. The period duration (h) and the debt maturity are set to one quarter.

Table 4: Simulation results of CRR and Poisson with redemption and recovery

	D/Y (quarterly)	Default probability (annual)	$\sigma(C)$	$\sigma(Y)$	$\sigma(TB/Y)$	$\rho(TB/Y, Y)$
CRR	40.4%	0.0%	2.9%	2.5%	0.8%	−37.3%
Poisson	37.8%	3.7%	1.6%	1.0%	1.4%	+7.1%

Quarterly frequency. $\sigma(X)$ is the standard deviation of X, $\rho(X, X')$ is the correlation of X with X'. TB is the trade balance. The solution to the detrended model is approximated using value function iteration on a grid of 40 points for the debt-to-GDP ratio. Second order moments are computed after applying an HP filter with frequency parameter 1600, over 500 simulations of 1000 periods each (simulations start with no debt and the first 15 periods of each simulation are dropped).

Table 4 reports the moments of the models when simulated for quarterly decisions. We now see a striking feature of each model: while the CRR model is unable to simulate a proper default rate, it does replicate the proper correlation of the trade balance with output. The intuition is the following. In the CRR case, output goes up or down smoothly. Being credit constrained, the country borrows more in good times and less in bad times so that consumption tracks output correspondingly, while the debt-to-GDP is maintained constant, at its upper limit. The Poisson case, instead, displays essentially no correlation between output and the current account.

5.3. The maturity of debt

In the theoretical exercise of the previous section, the time horizon and the maturity of the debt are shrinking simultaneously. One would want to separate them to see their respective contributions to the result presented above. To that end, we modify the model

[7] Note that in our framework the two options are not strictly equivalent because, in the case of a high V, the country will repeatedly pay the negative growth shock that occurs after a default. In the $x = 0$ case, the country pays the negative growth shock only once.

described in section 3 by disconnecting the maturity of the bonds from the discretization step (h). Following the Hatchondo and Martinez (2009) approach, we now assume that bonds are perpetual, and promise an infinite stream of coupons which decrease geometrically at a constant instantaneous rate δ. A bond emitted at period t promises to pay 1 unit of good in period $t+h$, and $e^{-\delta(k-1)h}$ units at period $t+kh$ for $h \geq 2$. The Macaulay definition of the duration of such a bond, in a riskless world, is $\frac{1+r}{\delta+r}$. Also note that, in a riskless world, the net present value of such a bond from the point of view of period t is $\frac{1}{\delta+r}$.

Table 5 reports the moments of the model when the maturity is held constant while the discretization step h varies. It is the analog of Table 3, using the same model and calibration, except for the maturity which is held constant at one year across simulations.

Table 5: Moments of discretized Lévy processes for various period durations but constant maturity

Period duration (h, in quarters)	4	2	1
Discretized Brownian process			
Default threshold (debt-to-GDP, quarterly, %)	48.4	52.9	70.9
Default probability in 10 years (%)	34.2	0.0	0.0
Discretized Poisson process			
Default threshold (debt-to-GDP, quarterly, %)	48.4	50.0	50.7
Default probability in 10 years (%)	33.8	34.4	34.3

The processes are parameterized as described in section 5.1, except for the maturity set at 1 year (in the Macaulay sense). The solution to the detrended model is approximated using value function iteration on a grid of 40 points for the debt-to-GDP ratio. Moments are obtained by averaging over 5,000 simulations of a length of 10 years, starting with no debt.

One can observe that Table 5 is very similar to Table 3.[8] This shows that the results obtained in sections 4 and 5.1, about the intrinsic differences between Brownian and Poisson processes in relation to default events, are not an artifact generated by the assumption that the discretization step equals the maturity of the bond. Even when maturity is held constant, the discretization step—which in our setup corresponds to the frequency of decision making—matters, and as one approaches continuous time decision making, the Brownian and Poisson lead to very different outcomes.

In the simulations that we have presented so far, the discretized Brownian process is not able to produce defaults, even when the discretization step goes to zero and the maturity is held constant. This does not mean however that proposition 2 can be generalized to longer maturities; indeed, papers such as Nuño and Thomas (2015) demonstrate that default can occur in continuous time models of strategic default with long maturities, even when GDP follows a Brownian process. In order to analyze the sensitivity of our results to debt maturity, Table 6 reports simulations results where the discretization step is held constant but maturity

[8] Note that we omitted the cases $h = 0.33$ and $h = 0.08$, because we could not obtain convergence of the model with constant maturity for such small values.

varies (the opposite of Table 5).

Table 6: Moments of discretized Brownian process for various maturities but constant period duration

Debt maturity (Macaulay duration, in quarters)	1	4	20	40
Default threshold (debt-to-GDP, quarterly, %)	68.6	70.0	44.8	54.6
Default probability in 10 years (%)	0.0	0.0	0.0	89.3
Welfare (starting from no debt)	−4.357	−4.357	−4.544	−4.606

The processes are parameterized as described in section 5.1, for a period duration (h) of one quarter. The solution to the detrended model is approximated using value function iteration on a grid of 40 points for the debt-to-GDP ratio. Moments are obtained by averaging over 5,000 simulations of a length of 10 years, starting with no debt.

In this numerical example based on the discretized Brownian case, where the period duration (h) is set to one quarter, defaults appear when the debt maturity reaches 10 years. Our framework and results are therefore consistent with the possibility of defaults for Brownian GDP and long maturities, as in Nuño and Thomas (2015). However, (i) the maturity needs to be large for default to materialize, and (ii) long maturities do not seem to be optimal. This is apparent in Nuño and Thomas (2015, Table 3, p. 34, "No inflation" case): reducing bond duration increases welfare and increases time to default (*i.e.* reduces default probability). Similarly, in the present paper, what Table 6 shows is that, as maturity increases and approaches the point at which defaults appear, welfare markedly decreases.

Table 6 report welfare for a few duration values (as Nuño and Thomas, 2015 do). In general, we should expect welfare to be non-monotonic as a function a duration. The main trade off here is that long maturities expose creditors to dilution risk, but maturities that are too short expose the country to refinancing risk. Both dilution risk and refinancing risk can increase the cost of debt and/or reduce a country's debt capacity. Hatchondo et al. (2016) analyze this trade-off in order to study a government's optimal choice of maturity.

Our analysis captures these effects: Table 6 indeed suggests that long maturities are not optimal, as mentioned above.[9] Moreover, the next section (where we return to the case where maturity and discretization step are the same for tractability) provides intuition for why very small maturities drastically reduce a country's debt capacity and that this can be seen as a consequence of an exacerbated refinancing risk problem.[10] This refinancing risk only becomes an issue for very short time windows. For larger time intervals, as Hatchondo et al. (2016, p. 1388) acknowledge, "[...] the government may choose to carry lower debt levels in order to mitigate the higher rollover risk implied by issuing only one-period bonds."

[9] Table 6 indicates that there is no default for a duration of 20 quarters. Hence, there is no dilution along the equilibrium path. This does not mean, however, that dilution risk does not play a role: the default threshold is much lower than in columns 1 and 2, which can be due to the *possibility* of dilution—we thank one referee for pointing this out.

[10] All other things equal, a low debt capacity harms welfare because the government is impatient and wants to borrow.

And indeed, in the CRR case, the government lowers debt when approaching the risky region, thereby efficiently monitoring a significant amount of debt and avoiding default.

Obviously, our analysis is not about solving an optimal maturity problem. For such a study, see Hatchondo et al. (2016), in which an exogenous endowment process is specified as an AR(1) (in particular shocks to $\log Y_t$ are Gaussian). An interesting question for future research is to see how their results would be modified with another endowment process, *e.g.* with the introduction of Poisson risk.

6. A further look at the limiting case of continuous time

As shown in Table 3, the optimal debt-to-output ratio is falling down when the time horizon h goes below one week. In this section we explain why this is so in the context of the CRR model and under an alternative approximation of the Brownian process, and reinterpret it in a pure continuous-time version of our analysis.

6.1. The CRR case

The following proposition shows that the maximum level of debt falls down to zero in the CRR case as the time interval h becomes very small.

Proposition 4. *Consider the CRR specification with time interval h. The critical annualized debt-to-output ratio goes to zero as h goes to zero.*

Proof. From Proposition 2, we know that, in the continuation (*i.e.* non-default) region, debt is perfectly risk-free in the CRR case. The non-negative consumption borrowing constraint (see discussion following the proof) therefore applies:

$$\begin{aligned} D &\leq \sum_{k\geq 0} e^{-rkh} Y e^{-k\sigma\sqrt{h}} \\ &\leq \frac{Y}{1-e^{-rh-\sigma\sqrt{h}}}. \end{aligned}$$

The critical annualized debt-to-output ratio $hd^*(h)$ therefore satisfies:

$$h\, d^*(h) \leq \frac{h}{1-e^{-rh-\sigma\sqrt{h}}},$$

which goes to 0 as $h \to 0$. □

As debt is risk-free, the country must be able to repay it even under the worst scenario and is therefore subject to a debt limit, the *non-negative consumption borrowing constraint*, using the terminology of Zhang (1997). This constraint, studied *e.g.* in Aiyagari (1994), states that the country can never have more debt than the discounted sum of outputs over the worst output path. Of course, in our framework, this constraint arises endogenously. But it implies that if the present value of future output can drop to zero "quickly" with positive probability, no borrowing is possible. "Quickly" here stems from the fact that GDP is subject to shocks whose volatility is of an order of magnitude proportional to $\sqrt{h}$. Yet the ability of the country to control its debt is proportionate to h (by modifying the consumption path). When the shocks to the economy become too frequent, which is what happens when h falls down to zero, the country is unable to monitor its debt-to-GDP ratio. Unmonitorable negative shocks prevent lending in the first place.

6.2. The Gaussian case

What happens if we do not consider the CRR process, but rather another approximation of the Brownian motion? We simulate the model under Gaussian shocks to $\log Y_t$: assume that the law of motion of output is given by:

$$\log(Y_{t+h}) - \log(Y_t) \sim N(\mu h, \sigma^2 h).$$

This formulation entails that output may go down to near zero in any finite time horizon.

Table 7: Simulation results for the Gaussian discrete approximation

Period duration (h, in quarters)	4	2	1	0.33	0.08	0.01
Default threshold (debt-to-GDP, quarterly, %)	28.8	29.0	32.2	41.3	45.0	4.0
Default probability in 10 years (%)	8.6	6.3	5.2	2.6	$\simeq 0$	$\simeq 0$

Table 7 numerically illustrates the implications of our model under this scenario. We do find that d^* drops to zero for very small h, as in the CRR case. Importantly, Table 7 also confirms the robustness and the insights of Proposition 2. Indeed, under the Gaussian model, the risk of default, although non-zero (due to the unbounded support of the Gaussian distribution), is still extremely small.

The comparison between Table 7 and Table 3 is very telling. Since both approximations converge to the same continuous process, the same equilibrium property in the limit—the impossibility to borrow—holds. There is no reason however to expect that default thresholds should agree off the limit. Actually, the spread out nature of the output distribution makes the country behave more prudently for high values of h: as a consequence, default thresholds are lower compared to the CRR case, and default probabilities are small but never exactly zero—because there is always a chance that output realizes at an arbitrarily low value next period.

6.3. A continuous time smooth economy

As mentioned above, the fact that no lending takes place for very small time windows is due to the theoretical possibility that GDP quickly falls down to almost zero. This scenario is very unlikely (while this would certainly be possible for an asset) but has a huge impact on a country's borrowing opportunities. We now restate the problem in a pure continuous-time framework, considering a more realistic assumption on output.

Start from the following specification of the output process (Y_t):

$$Y_t = X_t + Q_t,$$

where X is a positive diffusion:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

and Q is a deterministic non-decreasing process; Q_t represents the minimal flow of output available at time t. The preferences described in section 3 are modified in a straightforward manner:

$$J^*(D_t, Y_t) = \max_{(C_{t+s})_{s\geq 0}} \mathbb{E}_t \left[\int_t^\infty e^{-\rho s} u(C_{t+s}) ds \right].$$

Similar adjustments apply to the definitions of the value functions J^r and J^d. The stock of debt evolves according to

$$dD_t = \Pi(t, D_t, X_t, Q_t)D_t dt + (C_t - Y_t)dt,$$

where C_t is aggregate consumption and $\Pi(t, D_t, X_t, Q_t)$ is the risk-adjusted interest rate. The default threshold is denoted by $D^*(t, X_t, Q_t)$, assumed to be itself a diffusion. A version of this model in which the risk-adjusted interest rate is set in an *ad hoc* way is explicitly solved in Cohen (1993). The problem however is precisely that, in general, the interest rate cannot be solved for explicitly. Nonetheless, in the limit of continuous-time, the default time associated with a diffusion has no intensity, and zero-maturity debt spreads must be zero:

$$\lim_{h\to 0} \frac{1}{h} \mathbb{P}(\exists t < h, D_t > D^*(t, X_t, Q_t) | D_0 < D^*(0, X_0, Q_0)) = 0.$$

If lenders are subject to competition, they can only require the risk-free rate: $\Pi = r$ in the continuation region. For this to be consistent with equilibrium, debt must be effectively risk-free. What does it imply in terms of debt limits? To fix ideas, let X be a geometric Brownian motion: $\mu(t, X_t) = \mu X_t$ and $\sigma(t, X_t) = \sigma X_t$ and Q have constant growth: $Q_t = Q_0 e^{\nu t}$ with $\nu < r$.[11] In that case, the default threshold is a homogeneous function $D^*(X, Q)$. Assuming D^* of class C^2, $t \mapsto D^*(X_t, Q_t)$ is indeed a diffusion due to Itô formula, and the previous observation applies: debt must be risk-free.

Let's define:

$$\underline{D}(Q) = \frac{\lambda Q}{r - \nu}.$$

Recall that λ is the level of the penalty imposed to the country in case of default. $\underline{D}(Q)$ represents the present value of deterministic output forfeited upon default. With this definition in mind, we can state the following result:

Proposition 5. *In this continuous-time economy with vanishing debt maturity, debt is risk-free, and the country can sustain a positive level of debt; the default threshold $D^*(X, Q)$ satisfies $D^*(X, Q) = \underline{D}(Q)$ for all $X > 0$.*

[11] This specification preserves the homothetic and stationarity properties and therefore simplifies the exposition.

Proof. See appendix Appendix C. □

An important step of the proof is to verify that $\underline{D}(Q)$ is the default threshold in the deterministic economy ($X = 0$). This is done by working out the implications of endogenous debt limits, the *no default borrowing constraints*, in the terminology of Zhang (1997): for a debt contract to be feasible at the risk-free rate, the borrower must have no incentive, at any point in time, to deviate towards default. These constraints appear as soon as one focuses on default-free debt contracts (see *e.g.* Bai and Zhang, 2010). Of course, in a deterministic economy, debt has to be risk-free, and the no default borrowing constraints therefore apply.

The proof is interesting *per se* as it relies on a new lemma that allows to compare the valuations of cash flows under different discount rates (as is the case here, since the government discounts the future at a rate higher than the world risk-free rate). This provides an elegant characterization of the debt capacity in a deterministic economy.

Notice that the special case $Q = 0$ provides a no-lending result ($D^*(X, 0) = 0$) similar to the one obtained in section 6.1. When $Q > 0$, however, the country can sustain a positive debt level. And the addition of a smooth, risky process to the deterministic component of output does not change the debt capacity of the borrowing country, nor does it alter the risk-free nature of debt that holds in the deterministic case.

The only difference between the CRR model of section 4 and the limiting case studied in this section is the ability of a country to manage refinancing risk. In the limiting case, debt must be locally risk-free, yet there is a positive probability to reach the default frontier. This possibility propagates back into a no-lending result (in the sense that the Brownian part of output does not increase debt capacity, Proposition 5). This effect is by no means a peculiarity of our framework. There is in fact an analogy with the studies of refinancing risk in the context of dynamic debt runs: see *e.g.* Acharya et al. (2011). They show how high rollover frequencies can negatively impact a bank's debt capacity. The mechanism is similar: if the default threshold is large, debt can't be paid back at once and must therefore be refinanced. But close to the threshold, risk is very high, and there is no suitable interest rate to compensate lenders. This reduces the previous period's debt capacity, and so on until a debt level is reached that can be paid back without relying on refinancing (typically the liquidation value in this literature, or the deterministic component of output as in Proposition 5).

6.4. Comparison with an economy subject to a Poisson shock

Instead of smooth shocks to output, consider the following (theoretical) dynamics:

$$Y_t = Q_t \mathbb{I}_{\{t<\tau\}},$$

where Q_t is defined as before, $\mathbb{I}$ is the indicator function and τ is exponentially distributed with parameter η. Y grows deterministically until a Poisson shock puts the output to zero forever (assume $0 < \gamma < 1$ to have well-defined utilities when $C_t = 0$).

Unless it has zero debt, a country will always default at τ. Two options are available: (i) the country does not take on any debt, or (ii) it does, and the economy behaves as in the deterministic case of the previous section (case $X = 0$, see appendix Appendix C) until default at τ, when the instantaneous interest rate r is replaced by $r + \eta$.

Option (i) is clearly not optimal for an impatient government. The country therefore accepts a risk of default (which, indeed, happens with probability 1), in effect "ignoring" this possibility by acting as in the deterministic case with no Poisson shock under an instantaneous interest rate $r + \eta$. The country's strategy under a Poisson shock is therefore the opposite as the one under smooth shocks: in the former case, debt is risky and the government does not always try to avoid default; in the latter case, debt is risk-free, and the government (if possible, as in our CRR specification of the model) monitors the stock of debt by repaying debt as the default threshold approaches.

6.5. A summary

In the CRR case as in the smooth continuous case model, a country always finds it preferable to monitor its debt in order to remain away from the risky region. This is opposite to the Poisson case, where the country embarks into a risky strategy. From this theoretical perspective, the result obtained in the CRR case is robust to the continuous time case: the only equilibrium in the CRR/Brownian case is one without default.

7. Conclusion

This paper has presented a sovereign debt model inspired by a strand of the corporate finance literature, in which a smooth and a jump process are distinguished. We have shown that the default properties of Eaton-Gersovitz models depend critically on the choice of the output process. Poisson shocks are needed to account for the risk of default while smooth (monitorable) shocks are necessary to obtain the counter-cyclicality of the current account.

Finally, as we have shown, GDP may come down significantly without triggering a default, if the process that brings it down is a sequence of bad shocks of small magnitude, that the country can adjust to. This could be one factor explaining the weak correlation between output gap and default that is documented by Tomz and Wright (2007) and that standard models of sovereign default are unable to replicate. We leave the exploration of this conjecture for further research.

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Appendix

Appendix A. CRR & Poisson cases

Appendix A.1. General case

Lemma 6 (Eaton and Gersovitz, 1981)**.** *Default incentives are stronger the higher the debt.*

Lemma 7. *Default occurs if and only if the debt-to-GDP ratio is higher than a given threshold d^*.*

Proof. This is a straightforward implication of the isoelasticity of preferences and of the Markovian and homothetic nature of the stochastic process driving output and recovery. □

Lemma 8. *The country never chooses an indebtedness level such that default is sure tomorrow (*i.e.* a level for which $\tilde{L}'(s, \tilde{D}'(s)) = 0$).*

Proof. If the country was choosing an indebtedness level such that default was sure tomorrow, it would raise nothing today on the markets ($\tilde{L}'(s, \tilde{D}'(s)) = 0$). It is clear that choosing a zero indebtedness level tomorrow is a better strategy: it gives the same consumption today, and it gives a higher utility tomorrow (because the value function is decreasing in debt). □

Appendix A.2. CRR case

In this section, p denotes the probability that output is low tomorrow (*i.e.* $p = \frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{h}$). In order to demonstrate proposition 2, we begin by establishing the following lemma:

Lemma 9. *In the CRR case, if $h \leq \frac{1}{\left(4\sigma + \frac{\mu}{\sigma}\right)^2}$, the interest rate incorporating a risk premium does not occur in equilibrium.*

Proof. By contraposition. Assume that for some state s, the optimal choice is to repay and $\tilde{L}(s, \tilde{D}'(s))$ is equal to $e^{-rh}(1-p)\tilde{D}'(s)$. This is the risky case where the country will repay next period in the good state of nature, but default in the bad state (the other two possible values for $\tilde{L}(s, \tilde{D}'(s))$ are $e^{-rh}\tilde{D}'(s)$ and 0, since output can take only two values). This implies that:

$$\frac{\tilde{D}'(s)}{e^{\sigma\sqrt{h}}Y} \leq d^* < \frac{\tilde{D}'(s)}{e^{-\sigma\sqrt{h}}Y}.$$

We also have:

$$J^r(s) = u(Y - D + e^{-rh}(1-p)\tilde{D}'(s)) + (1-p)e^{-\rho h} J^r(\tilde{D}'(s), e^{\sigma\sqrt{h}}Y) + p\, e^{-\rho h} J^d(e^{-\sigma\sqrt{h}}Y).$$

Let $D'_2 = e^{-2\sigma\sqrt{h}}\tilde{D}'(s)$. We therefore have $\frac{D'_2}{e^{-\sigma\sqrt{h}}Y} \leq d^*$, which means that this level of indebtedness is in the safe zone and that $\tilde{L}(s, D'_2) = e^{-rh}D'_2$. If the country was choosing that level of indebtedness, it would get:

$$J^r_2(s) = u(Y - D + e^{-rh}D'_2) + (1-p)e^{-\rho h} J^r(D'_2, e^{\sigma\sqrt{h}}Y) + p\, e^{-\rho h} J^r(D'_2, e^{-\sigma\sqrt{h}}Y).$$

By optimality of $\tilde{D}'(s)$, we have $J^r(s) > J_2^r(s)$. And since we have $J^r(D_2', e^{\sigma\sqrt{h}}Y) > J^r(\tilde{D}'(s), e^{\sigma\sqrt{h}}Y)$ (by lemma 6) and $J^r(D_2', e^{-\sigma\sqrt{h}}Y) > J^d(e^{-\sigma\sqrt{h}}Y)$ (by construction of D_2'), this implies that $u(Y - D + e^{-rh}(1-p)\tilde{D}'(s)) > u(Y - D + e^{-rh}D_2')$. In turn, this implies that $(1-p) > e^{-2\sigma\sqrt{h}}$ or $\log(1-p) > -2\sigma\sqrt{h}$. Using the concavity of the logarithm, this implies $p < 2\sigma\sqrt{h}$, which is equivalent to $h > \frac{1}{\left(4\sigma+\frac{\mu}{\sigma}\right)^2}$. □

Proposition 2. Assume that the country decides to repay. Given that next period output can take only two values, three cases are possible for tomorrow:

1. the country will repay in both states,
2. the country will repay in the good state of nature and default in the bad state,
3. the country will default in both states.

The second and third cases are excluded by lemmas 8 and 9. By forward recursion, it is clear that the country will always repay in the future. □

Appendix A.3. Poisson case

Proposition 3. Let $h < 1$ (with the obvious condition that $1/h$ is an integer). Since growth shocks are independent, the probability that no jump has occurred at time $t+1$ (*i.e.* $Y_{t+1} = e^g Y_t$) is equal to $(e^{-p_0 h})^{1/h} = e^{-p_0}$. By immediate induction on times $t, t+h, \ldots, t+1$ and the lemmas of appendix Appendix A, this state of probability e^{-p_0} features the lowest possible debt-to-output ratio at time $t+1$. Therefore, by lemma 7, if there was default in this state, there would be default in all other states. In other words, default at $t+1$ would be already certain at time t, a contradiction with $\tilde{\delta}'(s_t) = 0$. Hence the probability of not having a default between dates t and $t+1$ is superior to e^{-p_0}.

There are cases in which the probability of not having a default between t and $t+1$ is exactly equal to e^{-p_0}. One such case is when the country totally ignores the future ($\rho = 0$). The default threshold (expressed as a debt-to-GDP ratio) is clearly $d^* = \lambda$. Since, by lemma 8, default never happens in the good state of nature, the country will always choose the maximum debt level conditional on not defaulting in the good state: $D_{t+h} = d^* e^{gh} Y_t = \lambda e^{gh} Y_t$. This means that the country will default in the bad state: indeed, in that case $Y_{t+h} = e^{gh} k \tilde{m}_t Y_t$ so that $D_{t+h} > d^* Y_{t+h}$ and the debt-to-output ratio exceeds the default threshold d^*. Therefore the probability of default at each period is equal to the probability of moving to the bad state: $1 - e^{-p_0 h}$. Using the independence of growth shocks, we conclude that the probability of not having a default between dates t and $t+1$ is equal to e^{-p_0} (using the independence of growth shocks between periods). Another case where the per period probability of default is exactly equal to e^{-p_0} is when the Poisson shock is too severe: when the distribution $\tilde{m}$ is concentrated on sufficiently low values and g is small, a Poisson shock would bring wealth below the amount of debt, triggering default. □

Appendix B. Policy, price and value functions of Section 4

In section 4, we have studied two polar cases: in the CRR specification, the two-state Markov chain of output approximates a Brownian process, while in the Poisson specification, it approaches a jump process. We showed that the properties of the equilibrium stand in sharp contrast. Here, we provide graphical representations of the equilibrium policy, price and value functions for both the CRR and the Poisson case. Those will provide additional illustrations of the differences between the two equilibria.

Figure B.1 depicts the value functions. Notice that they are decreasing since plotted as a function of the current debt-to-GDP ratio. Unsurprisingly, the value functions are concave up to the default point. At the default point, J remains constant at J^d, while J^r is below J^d (to focus on the curve J we have not plotted this section of J^r).

Figure B.2 represents the policy functions. Notice that D' is *next period*'s choice of debt, while Y is the current GDP level. With obvious notation, next period's debt-to-GDP ratio is given by $\frac{D'}{Y g^+}$ in case of a good shock and $\frac{D'}{Y g^-}$ in case of a bad shock. We plot the lines $y = g^+ x$ (dotted) and $y = g^- x$ (dashes and dots). We obtain a graphical representation of the dynamics of the equilibrium: starting from a debt choice on the y-axis, D'/Y, moving horizontally until crossing the dotted line and looking at the corresponding value on the x-axis provides next period's debt-to-GDP ratio in case of a good shock. Similarly, doing this procedure using the dashes and dots lines gives next period's debt-to-GDP in case of a bad shock. We then see that, in the CRR case, consistent with our no-default theorem (Proposition 2), the country never allows its debt to exceed a level which may entail default at the next period. Indeed, the plateau intersects the dashes and dots line precisely at the default threshold (materialized by the horizontal dashed line). When debt is high, the country always adjusts in order to stay in the risk-free zone and never takes more debt than prescribed by the plateau. In the right panel (Poisson case), the situation is different, consistent with Proposition 3. For low levels of debt, D'/Y is increasing in D/Y,[12] until it reaches a first plateau at $\frac{D'}{Y} \approx 0.3$. This first flat section has the same interpretation as in the CRR case: as can be seen graphically, debt is still risk-free: the country prefers to control its indebtedness in order to avoid paying a higher interest on its debt and risking default. In the Poisson case, however, the welfare cost of keeping debt this low (thereby foregoing a lot of consumption today) becomes eventually too high for high levels of debt: the country accepts to pay a higher bond price and to be at risk of default, in order to consume more today.

Figure B.3 provides additional intuition:[13] in the CRR case, entering the risky zone (small flat section around the default threshold) is extremely costly as the bond price increases sharply. In the Poisson case, since the shock is severe, but unlikely, the effect of entering the risky zone on the bond price is much less important. At some point, the country accepts the possibility of default to consume more today. Notice that the larger the magnitude of the bad shock, the more costly it is to avoid default with probability 1. When this magnitude is too large, the country simply ignores the bad shock and default can occur: this is the "Panglossian effect" described by Cohen and Villemot (2015).

[12]The steep section corresponds to a region where the objective functions are extremely flat around the maximum, making the argmax move very rapidly, but has no particular economic interpretation.

[13]The bond price is the inverse $1/q$ of the price function depicted in the Figure.

Figure B.1: **Value functions**, $h = 1$. Left panel: CRR case. Right panel: Poisson case.

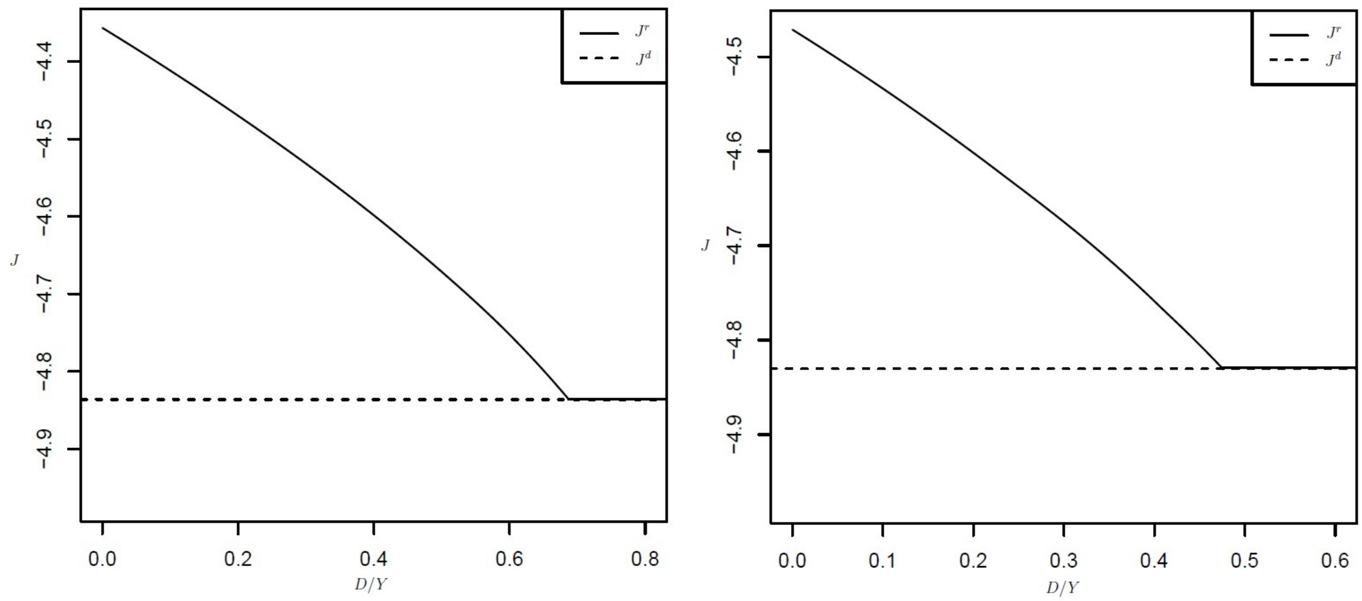


Figure B.2: **Policy functions**, $h = 1$. Left panel: CRR case. Right panel: Poisson case.

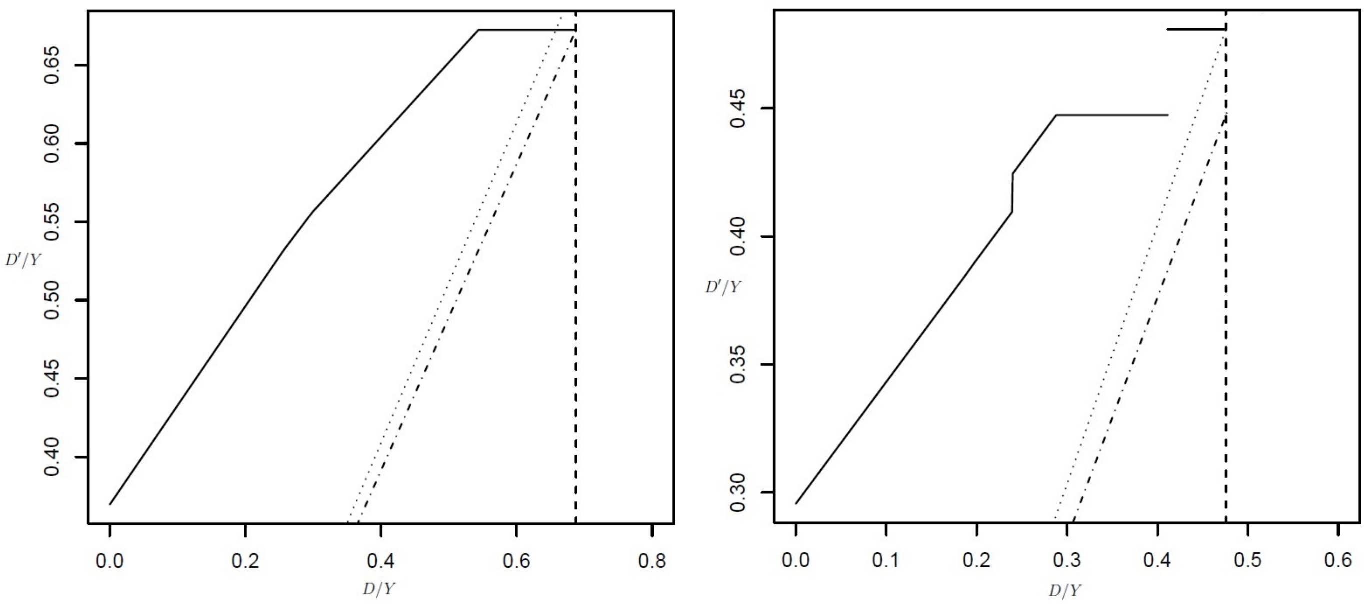
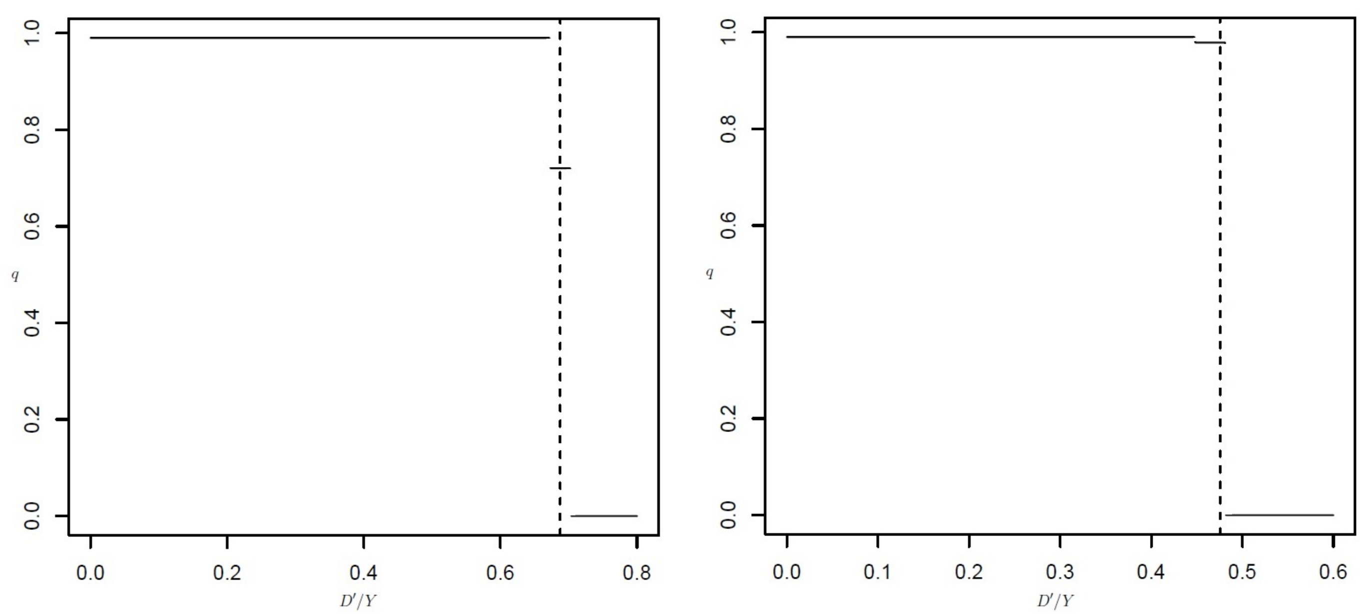


Figure B.3: **Price functions**, $h = 1$. Left panel: CRR case. Right panel: Poisson case.



Appendix C. Proof of proposition 5

Proof. Defaulting on debt reduces consumption by a factor $1-\lambda$. Since $X \geq 0$, the loss of consumption flow is at least λQ_t at each time t. Instead, the country could decide to repay λQ_t units of debt at each time. When $D < \underline{D}(Q)$, this strategy repays debt in finite time, and provides at least as much consumption before repayment and strictly more consumption after repayment. Therefore the country never defaults when $D < \underline{D}(Q)$ and $D^*(X, Q) \geq \underline{D}(Q)$.

We now show that there is in fact equality. Let us first consider the deterministic economy: $X = 0$. We begin by introducing a definition:

Definition Let $\alpha > 0$. We say that a flow (c_t) is α-preferred to (c'_t) at time s when

$$\int_s^\infty e^{-\alpha(t-s)} c_t dt \geq \int_s^\infty e^{-\alpha(t-s)} c'_t dt.$$

We will need the following lemma:

Lemma 10. *Let $\beta < \alpha$. If (c_t) is α-preferred to (c'_t) at all times, then (c_t) is β-preferred to (c'_t) at all times.*

Proof. By relabeling time 0, it is clearly sufficient to show that (c_t) is β-preferred to (c'_t) at time 0. The result then stems from the identity

$$\int_0^\infty e^{-\alpha t} c_t dt + (\alpha - \beta) \int_0^\infty e^{-\beta s} \int_s^\infty e^{-\alpha(t-s)} c_t dt ds = \int_0^\infty e^{-\beta t} c_t dt. \tag{C.1}$$

This means that the valuation under β is a linear combination with positive coefficients of the valuations under α. Therefore, if (c_t) is α-preferred to 0 at all times, it is also β-preferred to 0. The claim of the lemma follows by linearity. All is left is to establish (C.1):

$$\begin{aligned}
(\alpha-\beta)\int_0^\infty e^{-\beta s}\int_s^\infty e^{-\alpha(t-s)} c_t dt ds &= (\alpha-\beta)\int_0^\infty \int_0^\infty e^{-\beta s} e^{-\alpha u} c_{s+u} du ds \\
&= (\alpha-\beta)\int_0^\infty \int_0^t e^{-\beta s} e^{-\alpha(t-s)} c_t ds dt \\
&= (\alpha-\beta)\int_0^\infty e^{-\alpha t} c_t \int_0^t e^{(\alpha-\beta)s} ds dt \\
&= \int_0^\infty (e^{-\beta t} - e^{-\alpha t}) c_t dt.
\end{aligned}$$

□

In the deterministic economy with initial debt D, a consumption path (c_t) is feasible if it pays back the debt:

$$\int_0^\infty e^{-rt}(Q_t - c_t) dt = D, \tag{C.2}$$

and there is never an incentive for the country to deviate towards autarky: for all $s \geq 0$,

$$\int_s^\infty e^{-\rho(t-s)} u(c_t) dt \geq \int_s^\infty e^{-\rho(t-s)} u\left((1-\lambda)Q_t\right) dt. \tag{C.3}$$

With $Q_t = Qe^{\nu t}$ and $\tilde{c}_t = \frac{c_t}{Q_t}$, (C.3) rewrites: for all $s \geq 0$,

$$Q_s^{1-\gamma} \int_s^\infty e^{(-\rho+\nu(1-\gamma))(t-s)} u(\tilde{c}_t) dt \geq Q_s^{1-\gamma} \int_s^\infty e^{(-\rho+\nu(1-\gamma))(t-s)} u(1-\lambda) dt,$$

or, with $\alpha = \rho - \nu(1-\gamma)$:

$$\int_s^\infty e^{-\alpha(t-s)} u(\tilde{c}_t) dt \geq \int_s^\infty e^{-\alpha(t-s)} u(1-\lambda) dt.$$

Now, using Jensen's inequality for the measure $\alpha e^{-\alpha(t-s)} dt$ over $[s; +\infty[$ and the concave function u, we find that for all $s \geq 0$:

$$u\left(\alpha \int_s^\infty e^{-\alpha(t-s)} \tilde{c}_t dt\right) \geq u(1-\lambda).$$

Since u is increasing, we obtain that $(\tilde{c}_t)$ is α-preferred to $1-\lambda$ at all times. Since $\alpha > r - \nu$, Lemma 10 applies and we conclude that $(\tilde{c}_t)$ is $(r-\nu)$-preferred to $1-\lambda$ at time 0, *i.e.*:

$$\int_0^\infty e^{(\nu-r)t} (\tilde{c}_t - (1-\lambda)) dt \geq 0.$$

In terms of the non-normalized quantities c_t and Q_t, this means:

$$\begin{aligned} \int_0^\infty e^{-rt}(c_t - (1-\lambda)Q_t) dt &\geq 0, \\ \int_0^\infty e^{-rt}(c_t - Q_t) dt + \lambda \int_0^\infty e^{-rt} Q_t dt &\geq 0, \\ \frac{\lambda Q}{r-\nu} &\geq D, \end{aligned}$$

where the last line is due to (C.2). This establishes that for a debt level D to be compatible with equilibrium, we must have $D \leq \frac{\lambda Q}{r-\nu} = \underline{D}(Q)$.

We have established that $D^*(0, Q) = \underline{D}(Q)$. We now go back to the general case where X is not zero. If $D > \underline{D}(Q)$, then with positive probability X will drop arbitrarily close to 0, in which case the country will default (as D^* is clearly continuous). This is incompatible with the risk-free nature of debt. Therefore $D^*(X, Q) = \underline{D}(Q)$. □