Deterministic Models

Perfect foresight, nonlinearities and occasionally binding constraints

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Introduction

- Perfect foresight = agents perfectly anticipate all future shocks
- Concretely, at period 1:
  - agents learn the value of all future shocks;
  - since there is shared knowledge of the model and of future shocks, agents can compute their optimal plans for all future periods;
  - optimal plans are not adjusted in periods 2 and later
    $\Rightarrow$ the model behaves as if it were deterministic.
- Cost of this approach: the effect of future uncertainty is not taken into account (*e.g.* no precautionary motive)
- Advantage: numerical solution can be computed exactly (up to rounding errors), contrarily to perturbation or global solution methods for rational expectations models
- In particular, nonlinearities fully taken into account (*e.g.* occasionally binding constraints)

Outline

Outline

The (deterministic) neoclassical growth model

$$\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t.

$$c_t + k_t = A_t k_{t-1}^{\alpha} + (1-\delta) k_{t-1}$$

First order conditions:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left(\alpha A_{t+1} k_t^{\alpha-1} + 1 - \delta \right)$$
$$c_t + k_t = A_t k_{t-1}^{\alpha} + (1-\delta) k_{t-1}$$

Steady state:

$$\bar{k} = \left(\frac{1 - \beta(1-\delta)}{\beta \alpha \bar{A}} \right)^{\frac{1}{\alpha - 1}}$$
$$\bar{c} = \bar{A} \bar{k}^{\alpha} - \delta \bar{k}$$

Note the absence of stochastic elements! No expectancy term, no probability distribution

Dynare code (1/3)

rcb_basic.mod

```
var c k;
varexo A;
parameters alpha beta gamma delta;

alpha=0.5;
beta=0.95;
gamma=0.5;
delta=0.02;

model;
  c + k = A*k(-1)^alpha + (1-delta)*k(-1);
  c^(-gamma) = beta*c(+1)^(-gamma)*(alpha*A(+1)*k^(alpha-1) + 1 - delta);
end;
```

Dynare code (2/3)

rcb_basic.mod

```
// Steady state (analytically solved)
initval;
  A = 1;
  k = ((1-beta*(1-delta))/(beta*alpha*A))^(1/(alpha-1));
  c = A*k^alpha-delta*k;
end;

// Check that this is indeed the steady state
steady;
```

Dynare code (3/3)

rcb_basic.mod

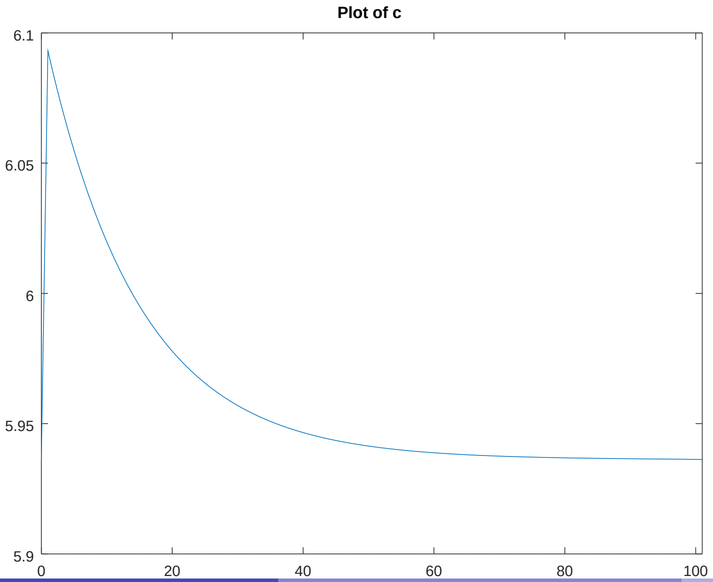
```
// Declare a positive technological shock in period 1
shocks;
  var A;
  periods 1;
  values 1.2;
end;

// Prepare the deterministic simulation over 100 periods
perfect_foresight_setup(periods=100);

// Perform the simulation
perfect_foresight_solver;

// Display the path of consumption
rplot c;
```

Simulated consumption path



The general problem

Deterministic, perfect foresight, case:

$$f(y_{t+1}, y_t, y_{t-1}, u_t) = 0$$

y	:	vector of endogenous variables
u	:	vector of exogenous shocks

Identification rule: as many endogenous (y) as equations (f)

Return to the neoclassical growth model

$$y_t = \begin{pmatrix} c_t \\ k_t \end{pmatrix}$$

$$u_t = A_t$$

$$f(y_{t+1}, y_t, y_{t-1}, u_t) = \begin{pmatrix} c_t^{-\sigma} - \beta c_{t+1}^{-\sigma} \left(\alpha A_{t+1} k_t^{\alpha-1} + 1 - \delta \right) \\ c_t + k_t - A_t k_{t-1}^{\alpha} + (1-\delta) k_{t-1} \end{pmatrix}$$

What if more than one lead or one lag?

- A model with more than one lead or lag can be transformed in the form with one lead and one lag using *auxiliary* variables
- Transformation done automatically by Dynare
- For example, if there is a variable with two leads x_{t+2}:
  - create a new auxiliary variable a
  - replace all occurrences of x_{t+2} by a_{t+1}
  - add a new equation: $a_t = x_{t+1}$
- Symmetric process for variables with more than one lag
- With future uncertainty, the transformation is more elaborate (but still possible) on variables with leads

Steady state

- A steady state, $\bar{y}$, for the model satisfies

$$f(\bar{y}, \bar{y}, \bar{y}, \bar{u}) = 0$$

- Note that a steady state is conditional to:
    - The steady state values of exogenous variables $\bar{u}$
    - The value of parameters (implicit in the above definition)
- Even for a given set of exogenous and parameter values, some (nonlinear) models have several steady states
- The steady state is computed by Dynare with the `steady` command
- That command internally uses a nonlinear solver

A two-boundary value problem

- Stacked system for a perfect foresight simulation over T periods:

$$\begin{cases} f(y_2, y_1, y_0, u_1) = 0 \\ f(y_3, y_2, y_1, u_2) = 0 \\ \vdots \\ f(y_{T+1}, y_T, y_{T-1}, u_T) = 0 \end{cases}$$

  for y_0 and y_{T+1} given.

- Compact representation:

$$F(Y) = 0$$

  where $Y = \begin{bmatrix} y_1' & y_2' & \dots & y_T' \end{bmatrix}'$
  and y_0, y_{T+1}, $u_1 \dots u_T$ are implicit

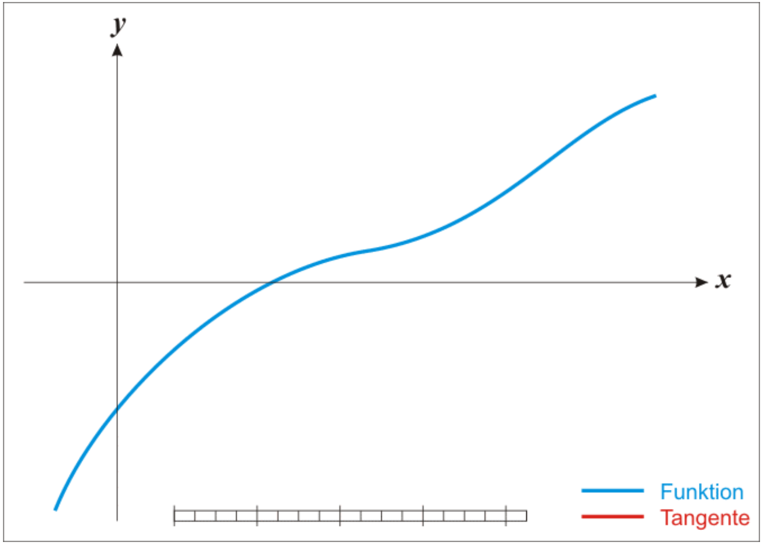
- Resolution uses a Newton-type method on the stacked system

Approximating infinite-horizon problems

- The above technique numerically computes trajectories for given shocks over a *finite* number of periods
- Suppose you are rather interested in solving an *infinite*-horizon problem
- One option consists in computing the recursive policy function (as with perturbation methods), but this is challenging
  - in the general case, this function is defined over an infinite-dimensional space (because all future shocks are state variables)
  - in the particular case of a return to equilibrium, the state-space is finite (starting from the date where all shocks are zero), but a projection method would still be needed
  - in any case, Dynare does not do that
- An easier way, in the case of a return to equilibrium, is to approximate the solution by a finite-horizon problem
  - consists in computing the trajectory with $y_{T+1} = \bar{y}$ and T large enough
  - drawback compared to the policy function approach: the solution is specific to a given sequence of shocks, and not generic

Outline

The Newton method (unidimensional)



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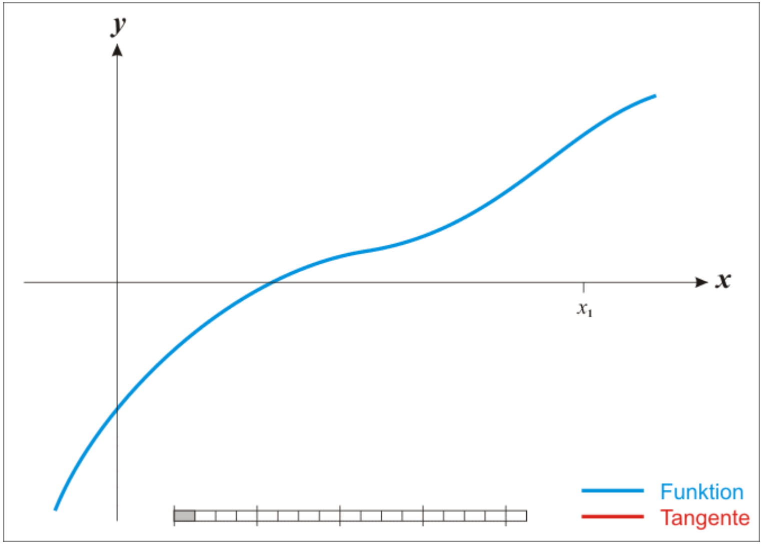

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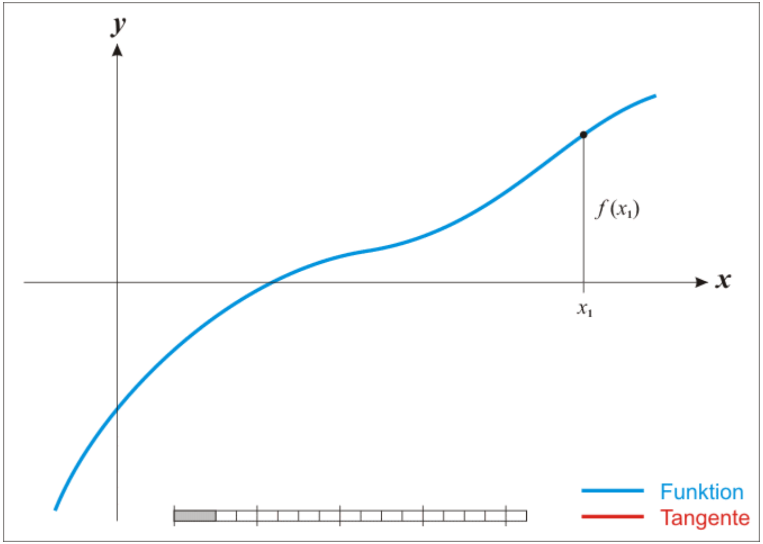

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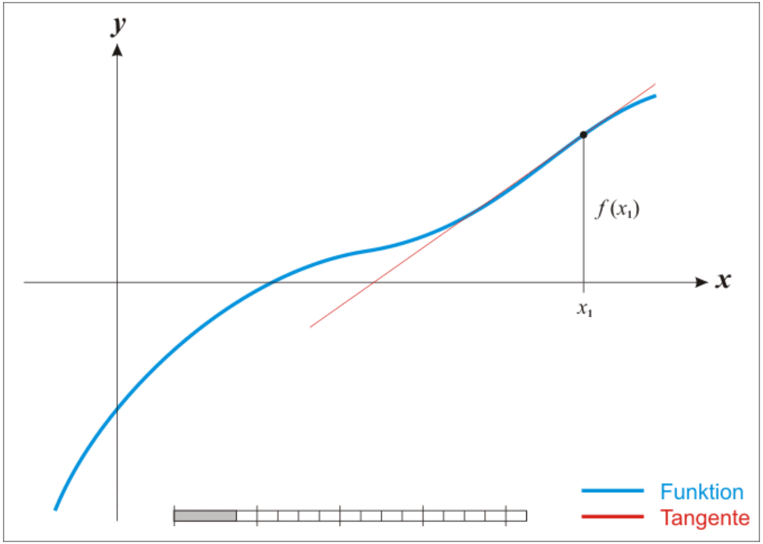

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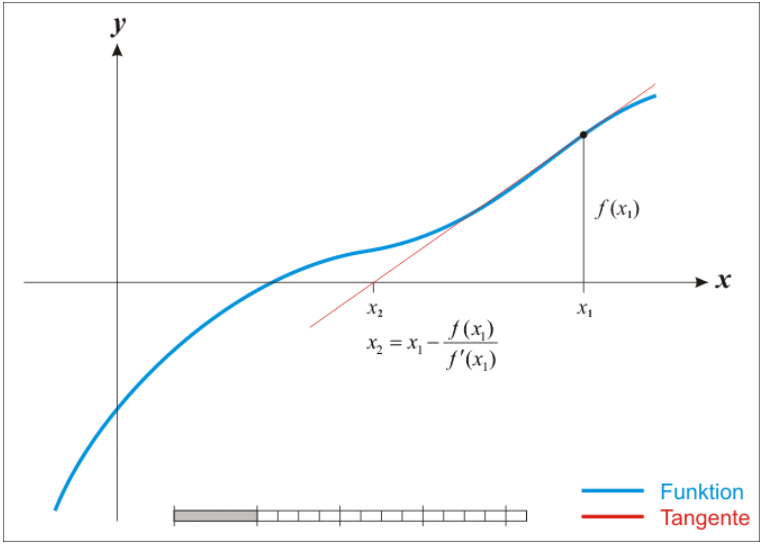

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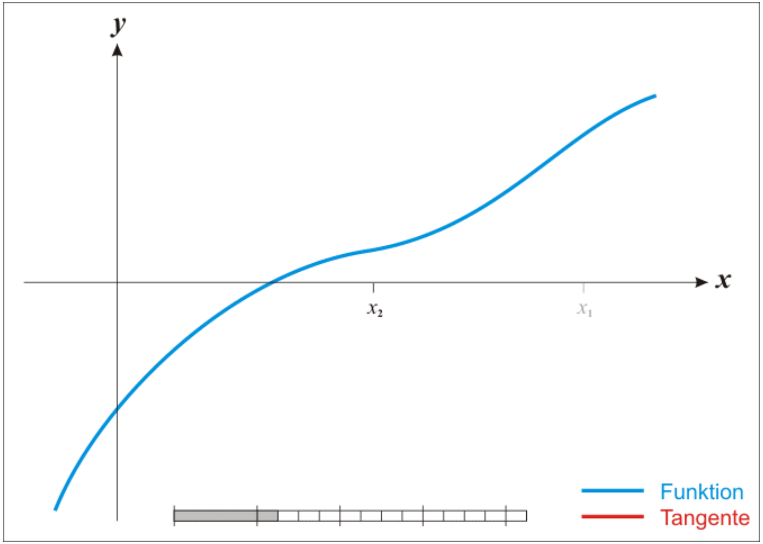

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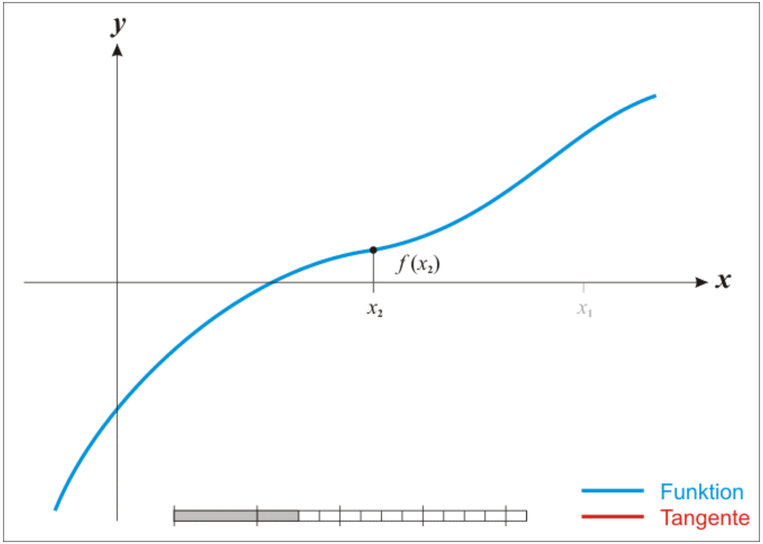

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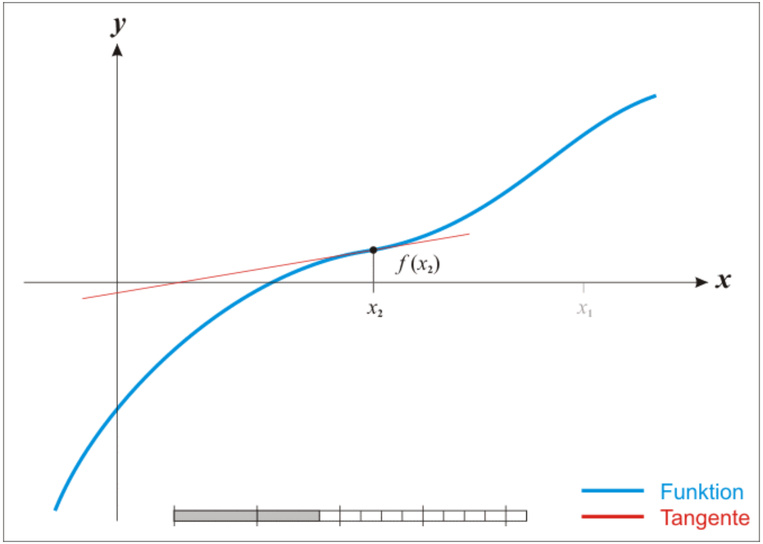

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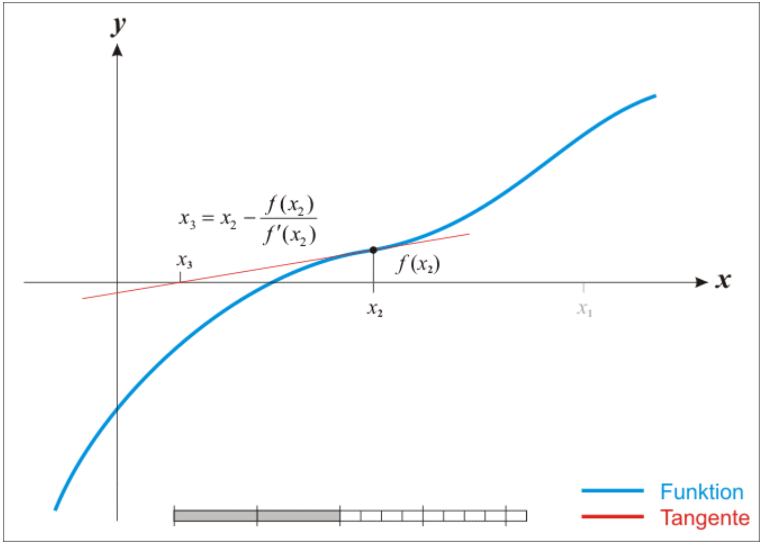

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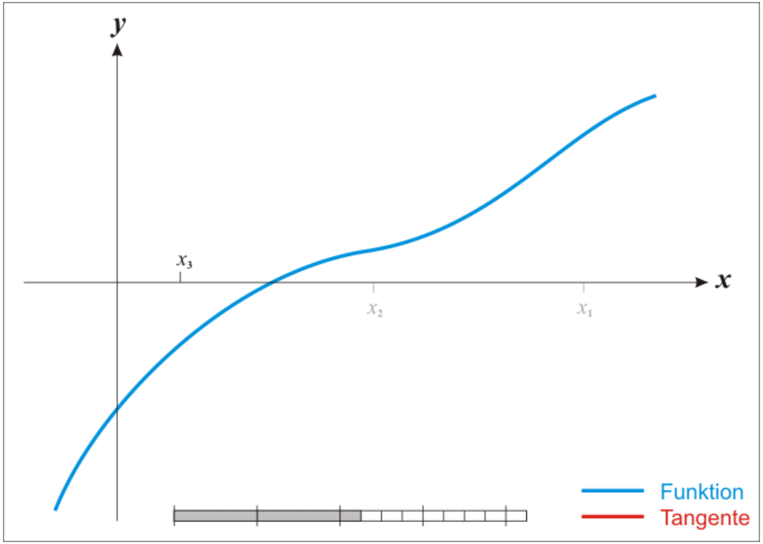

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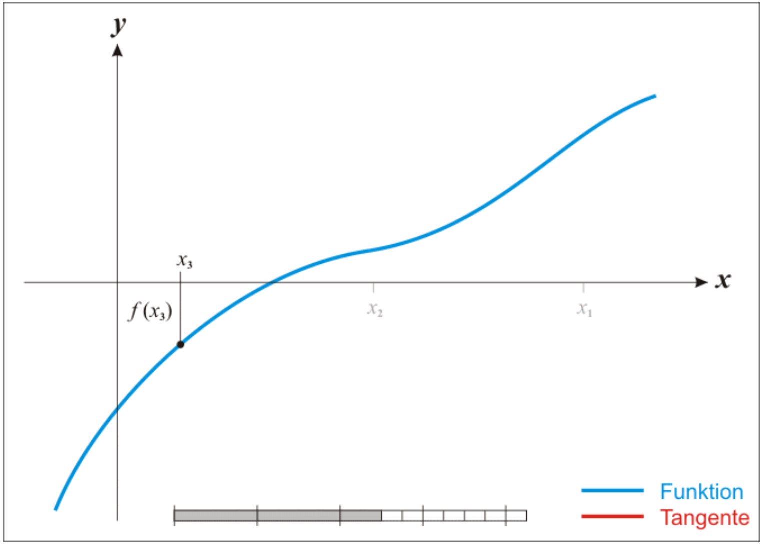

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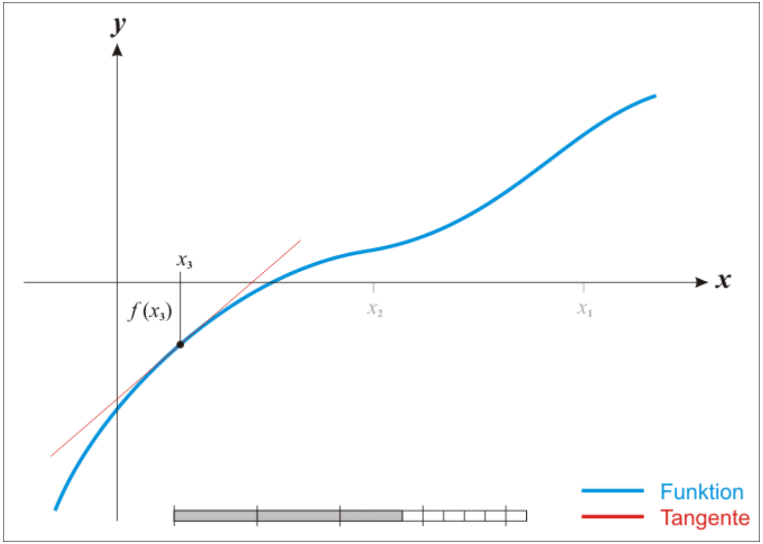

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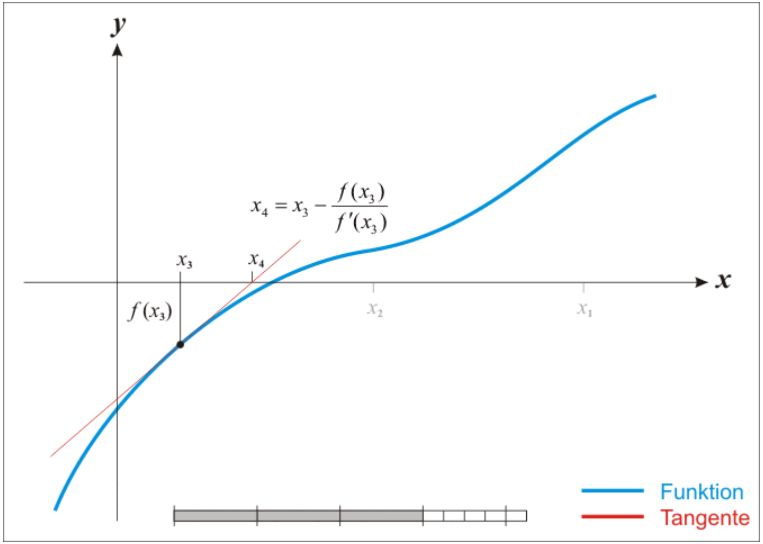

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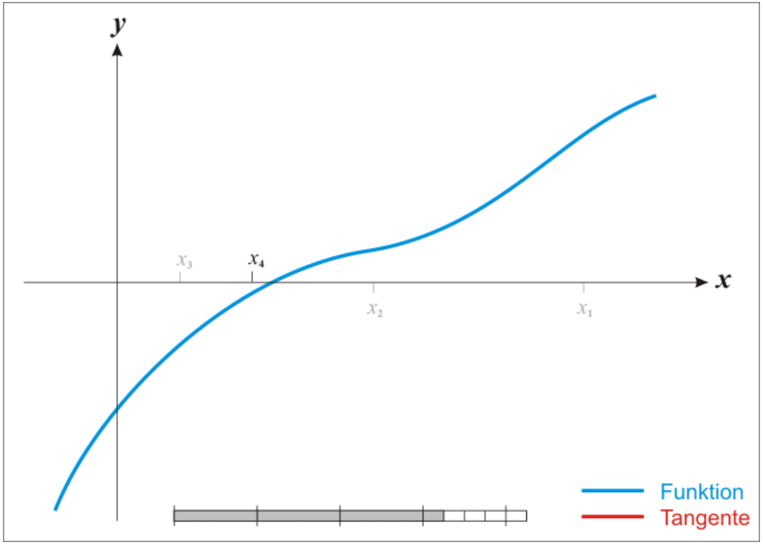

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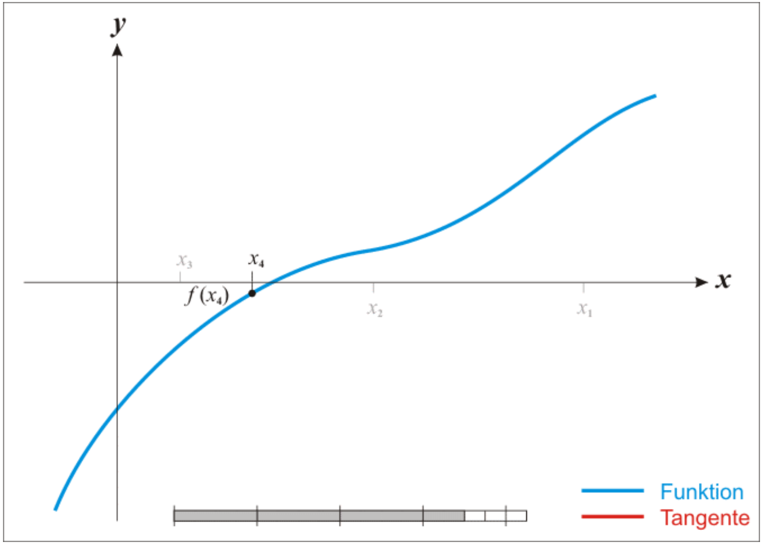

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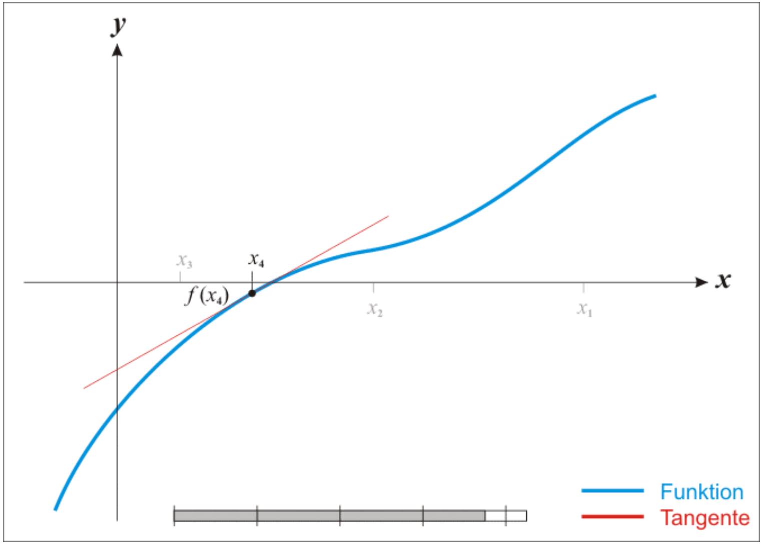

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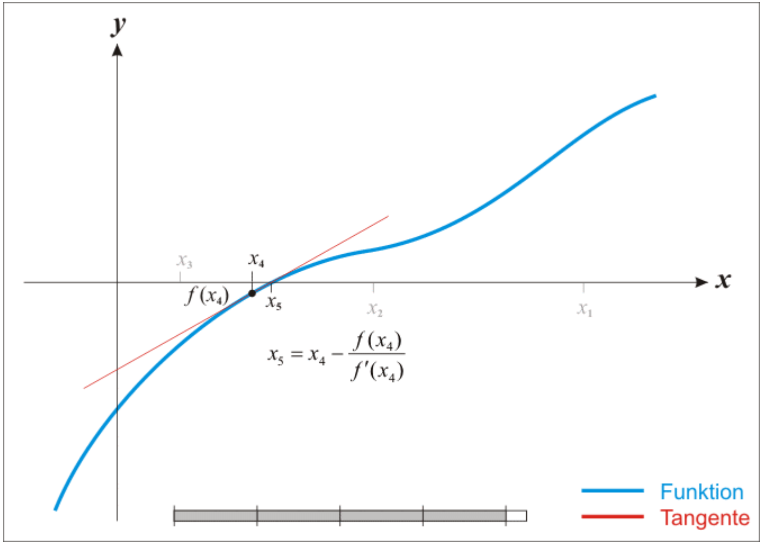

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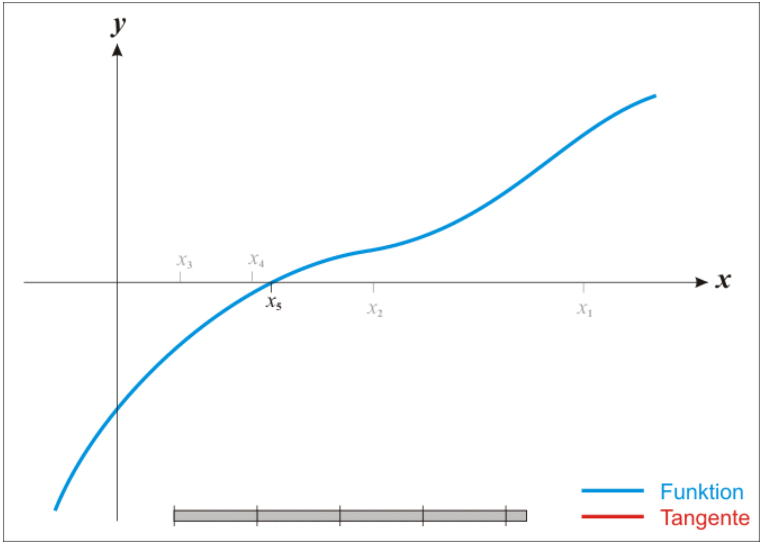


The Newton method (multidimensional)

- Start from an initial guess $Y^{(0)}$
- Iterate. Updated solutions $Y^{(k+1)}$ are obtained by solving a linear system:

$$F(Y^{(k)}) + \left[\frac{\partial F}{\partial Y}\right]\left(Y^{(k+1)} - Y^{(k)}\right) = 0$$

- Terminal condition:

$$||Y^{(k+1)} - Y^{(k)}|| < \varepsilon_Y$$

  or

$$||F(Y^{(k)})|| < \varepsilon_F$$

- Convergence may never happen if function is ill-behaved or initial guess $Y^{(0)}$ too far from a solution
  $\Rightarrow$ to avoid an infinite loop, abort after a given number of iterations

Controlling the Newton algorithm from Dynare

The following options to the `perfect_foresight_solver` can be used to control the Newton algorithm:

`maxit`	Maximum number of iterations before aborting (default: 50)
`tolf`	Convergence criterion based on function value (ε_F) (default: 10^{-5})
`tolx`	Convergence criterion based on change in the function argument (ε_Y) (default: 10^{-5})

`stack_solve_algo` select between the different flavors of Newton algorithms (see thereafter)

A practical difficulty

The Jacobian can be very large: for a simulation over T periods of a model with n endogenous variables, it is a matrix of dimension $nT \times nT$.

Three alternative ways of dealing with the large problem size:

- Exploit the particular structure of the Jacobian using a technique developped by Laffargue, Boucekkine and Juillard (was the default method in Dynare $\leq$ 4.2)
- Handle the Jacobian as one large, sparse, matrix (now the default method)
- Block decomposition, which is a divide-and-conquer method (can actually be combined with one of the previous two methods)

Shape of the Jacobian

$$\frac{\partial F}{\partial Y} = \begin{pmatrix} B_1 & C_1 & & & & & \\ A_2 & B_2 & C_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & A_t & B_t & C_t & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & A_{T-1} & B_{T-1} & C_{T-1} \\ & & & & & A_T & B_T \end{pmatrix}$$

where

$$A_s = \frac{\partial f}{\partial y_{t-1}}(y_{s+1}, y_s, y_{s-1}) \qquad B_s = \frac{\partial f}{\partial y_t}(y_{s+1}, y_s, y_{s-1})$$

$$C_s = \frac{\partial f}{\partial y_{t+1}}(y_{s+1}, y_s, y_{s-1})$$

Laffargue-Boucekkine-Juillard algorithm (1/5)

The idea is to triangularize the stacked system:

$$\begin{pmatrix} B_1 & C_1 & & & & \\ A_2 & B_2 & C_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{T-1} & B_{T-1} & C_{T-1} \\ & & & & A_T & B_T \end{pmatrix} \Delta Y = - \begin{pmatrix} f(y_2, y_1, y_0, u_1) \\ f(y_3, y_2, y_1, u_2) \\ \vdots \\ \vdots \\ f(y_T, y_{T-1}, y_T, u_{T-1}) \\ f(y_{T+1}, y_T, y_{T-1}, u_T) \end{pmatrix}$$

Laffargue-Boucekkine-Juillard algorithm (2/5)

First period is special:

$$\begin{pmatrix} I & D_1 & & & & \\ & B_2 - A_2 D_1 & C_2 & & & \\ & A_3 & B_3 & C_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{T-1} & B_{T-1} & C_{T-1} \\ & & & & A_T & B_T \end{pmatrix} \Delta Y = - \begin{pmatrix} d_1 \\ f(y_3, y_2, y_1, u_2) + A_2 d_1 \\ f(y_4, y_3, y_2, u_3) \\ \vdots \\ f(y_T, y_{T-1}, y_T, u_{T-1}) \\ f(y_{T+1}, y_T, y_{T-1}, u_T) \end{pmatrix}$$

where

- $D_1 = B_1^{-1} C_1$
- $d_1 = B_1^{-1} f(y_2, y_1, y_0, u_1)$

Laffargue-Boucekkine-Juillard algorithm (3/5)

Normal iteration:

$$\begin{pmatrix} I & D_1 & & & & \\ & I & D_2 & & & \\ & & B_3 - A_3 D_2 & C_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{T-1} & B_{T-1} & C_{T-1} \\ & & & & A_T & B_T \end{pmatrix} \Delta Y = - \begin{pmatrix} d_1 \\ d_2 \\ f(y_4, y_3, y_2, u_3) + A_3 d_2 \\ \vdots \\ f(y_T, y_{T-1}, y_T, u_{T-1}) \\ f(y_{T+1}, y_T, y_{T-1}, u_T) \end{pmatrix}$$

where

- $D_2 = (B_2 - A_2 D_1)^{-1} C_2$
- $d_2 = (B_2 - A_2 D_1)^{-1} (f(y_3, y_2, y_1, u_2) + A_2 d_1)$

Laffargue-Boucekkine-Juillard algorithm (4/5)

Final iteration:

$$\begin{pmatrix} I & D_1 & & & & \\ & I & D_2 & & & \\ & & I & D_3 & & \\ & & & \ddots & \ddots & \\ & & & & I & D_{T-1} \\ & & & & & I \end{pmatrix} \Delta Y = - \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{T-1} \\ d_T \end{pmatrix}$$

where

$$d_T = (B_T - A_T D_{T-1})^{-1} (f(y_{T+1}, y_T, y_{T-1}, u_T) + A_T d_{T-1})$$

Laffargue-Boucekkine-Juillard algorithm (5/5)

- The system is then solved by backward iteration:

$$
\begin{aligned}
y_T^{k+1} &= y_T^k - d_T \\
y_{T-1}^{k+1} &= y_{T-1}^k - d_{T-1} - D_{T-1}(y_T^{k+1} - y_T^k) \\
&\vdots \\
y_1^{k+1} &= y_1^k - d_1 - D_1(y_2^{k+1} - y_2^k)
\end{aligned}
$$

- No need to ever store the whole Jacobian: only the D_s and d_s have to be stored
- This technique is memory efficient (was the default method in Dynare $\leq$ 4.2 for this reason)
- Still available as option `stack_solve_algo=6` of `perfect_foresight_solver` command (or `stack_solve_algo=1` in Dynare 6)

Sparse matrices (1/3)

- Consider the following matrix with most elements equal to zero:

$$A = \begin{pmatrix} 0 & 0 & 2.5 \\ -3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Dense matrix storage (in column-major order) treats it as a one-dimensional array:

$$[0, -3, 0, 0, 0, 0, 2.5, 0, 0]$$

- Sparse matrix storage:
  - views it as a list of triplets (i, j, v) where (i, j) is a matrix coordinate and v a non-zero value
  - A would be stored as

$$\{(2, 1, -3), (1, 3, 2.5)\}$$

Sparse matrices (2/3)

- In the general case, given an $m \times n$ matrix with k non-zero elements:
  - dense matrix storage $= 8mn$ bytes
  - sparse matrix storage $= 16k$ bytes
  - sparse storage more memory-efficient as soon as $k < mn/2$

  (assuming 32-bit integers and 64-bit floating point numbers)
- In practice, sparse storage becomes interesting if $k \ll mn/2$, because linear algebra algorithms are vectorized

Sparse matrices (3/3)

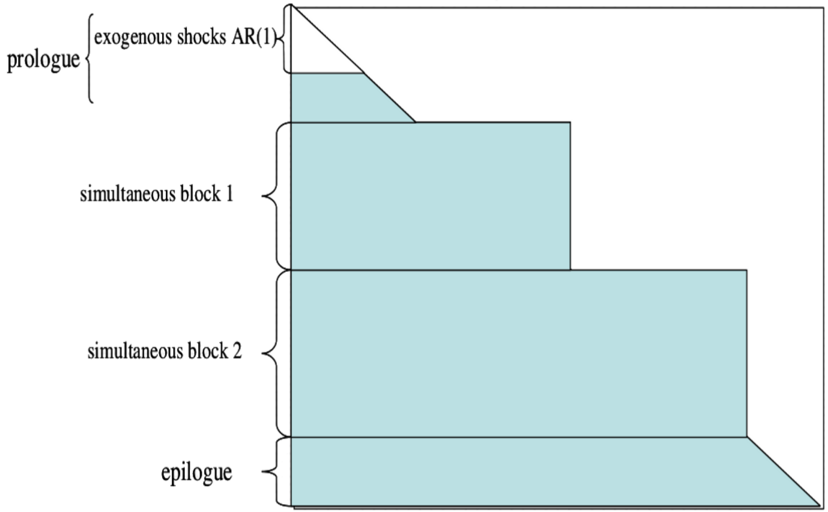
- The Jacobian of the deterministic problem is a sparse matrix:
  - Lots of zero blocks
  - The A_s, B_s and C_s usually are themselves sparse
- Family of optimized algorithms for sparse matrices (including matrix inversion for our Newton algorithm)
- Available as native objects in MATLAB/Octave (see the `sparse` command)
- Works well for medium size deterministic models
- Nowadays more efficient than Laffargue-Boucekkine-Juillard, even though it does not exploit the particular structure of the Jacobian
  $\Rightarrow$ now the default method in Dynare (`stack_solve_algo=0`)

Block decomposition (1/3)

- Idea: apply a divide-and-conquer technique to model simulation
- Principle: identify recursive and simultaneous blocks in the model structure
- First block (prologue): equations that only involve variables determined by previous equations; example: AR(1) processes
- Last block (epilogue): pure output/reporting equations
- In between: simultaneous blocks, that depend recursively on each other
- The identification of the blocks is performed through a matching between variables and equations (normalization), then a reordering of both

Block decomposition (2/3)

Form of the reordered Jacobian (equations in lines, variables in columns)



Block decomposition (3/3)

- Can provide a significant speed-up on large models
- Implemented in Dynare by Ferhat Mihoubi
- Available as option `block` to the `model` command
- Bigger gains when used in conjunction with `bytecode` or `use_dll` option
- Can be combined with any flavour of the Newton method applied to each individual block

Homotopy

- Another divide-and-conquer method, but in the shocks dimension
- Useful if shocks so large that convergence does not occur
- Idea: achieve convergence on smaller shock size, then use the result as initial guess for bigger shock size
- Algorithm:
  1. $\lambda \leftarrow 0$: scaling factor of shocks (simulation succeeds when $\lambda = 1$)
  2. $s \leftarrow 1$: step size
  3. Try to compute simulation with shocks scaling factor equal to $\lambda + s$, using last successful computation as initial guess (in the beginning, use initial steady state at all t as initial guess)
     - If success: $\lambda \leftarrow \lambda + s$. Stop if $\lambda = 1$. Otherwise possibly increase s.
     - If failure: diminish s.
  4. Go to 3
- Works with both temporary and permanent shocks (*i.e.* `shocks` and `endval`)
- Can be combined with any deterministic solver
- Used by default by `perfect_foresight_solver` (disable with option `no_homotopy`)

Outline

Example: neoclassical growth model with investment

The social planner problem is as follows:

$$\max_{\{c_{t+j},\ell_{t+j},k_{t+j}\}_{j=0}^{\infty}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, \ell_{t+j})$$

s.t.

$$\begin{aligned} y_t &= c_t + i_t \\ y_t &= A_t f(k_{t-1}, \ell_t) \\ k_t &= i_t + (1-\delta) k_{t-1} \\ A_t &= A^\star e^{a_t} \\ a_t &= \rho\, a_{t-1} + \varepsilon_t \end{aligned}$$

where ε_t is an exogenous shock.

Specifications

- Utility function:

$$u(c_t, \ell_t) = \frac{\left(c_t^{\theta}(1-\ell_t)^{1-\theta}\right)^{1-\tau}}{1-\tau}$$

- Production function:

$$f(k_{t-1}, \ell_t) = \left(\alpha k_{t-1}^{\psi} + (1-\alpha)\ell_t^{\psi}\right)^{\frac{1}{\psi}}$$

First order conditions

- Euler equation:

$$u_c(c_t, \ell_t) = \beta\, \mathbb{E}_t \left[u_c(c_{t+1}, \ell_{t+1}) \Big(A_{t+1} f_k(k_t, \ell_{t+1}) + 1 - \delta \Big) \right]$$

- Arbitrage between consumption and leisure:

$$\frac{u_\ell(c_t, \ell_t)}{u_c(c_t, \ell_t)} + A_t f_\ell(k_{t-1}, \ell_t) = 0$$

- Resource constraint:

$$c_t + k_t = A_t f(k_{t-1}, \ell_t) + (1 - \delta) k_{t-1}$$

Dynare code (1/3)

```
var k, y, L, c, A, a;
varexo epsilon;
parameters beta, theta, tau, alpha, psi, delta, rho, Astar;

beta = 0.99;
theta = 0.357;
tau = 2;
alpha = 0.45;
psi = -0.1;
delta = 0.02;
rho = 0.8;
Astar = 1;
```

Dynare code (2/3)

```
model;
  a = rho*a(-1) + epsilon;
  A = Astar*exp(a);
  y = A*(alpha*k(-1)^psi+(1-alpha)*L^psi)^(1/psi);
  k = y-c + (1-delta)*k(-1);
  (1-theta)/theta*c/(1-L) - (1-alpha)*(y/L)^(1-psi);
  (c^theta*(1-L)^(1-theta))^(1-tau)/c =
     beta*(c(+1)^theta*(1-L(+1))^(1-theta))^(1-tau)/c(+1)
      *(alpha*(y(+1)/k)^(1-psi)+1-delta);
end;
```

Dynare code (3/3)

```
steady_state_model;
  a = epsilon/(1-rho);
  A = Astar*exp(a);
  Output_per_unit_of_Capital=((1/beta-1+delta)/alpha)^(1/(1-psi));
  Consumption_per_unit_of_Capital=Output_per_unit_of_Capital-delta;
  Labour_per_unit_of_Capital=(((Output_per_unit_of_Capital/A)^psi-alpha)/(1-alpha))^(1/psi);
  Output_per_unit_of_Labour=Output_per_unit_of_Capital/Labour_per_unit_of_Capital;
  Consumption_per_unit_of_Labour=Consumption_per_unit_of_Capital/Labour_per_unit_of_Capital;

  % Compute steady state of the endogenous variables.
  L=1/(1+Consumption_per_unit_of_Labour/((1-alpha)*theta/(1-theta)*Output_per_unit_of_Labour^(1-psi)));
  c=Consumption_per_unit_of_Labour*L;
  k=L/Labour_per_unit_of_Capital;
  y=Output_per_unit_of_Capital*k;
end;
```

Scenario 1: Return to equilibrium

Return to equilibrium starting from $k_0 = 0.5\bar{k}$.

Fragment from `rbc_det1.mod`

```
...
steady;

ik = varlist_indices('k',M_.endo_names);
kstar = oo_.steady_state(ik);

histval;
  k(0) = kstar/2;
end;

perfect_foresight_setup(periods=300);
perfect_foresight_solver;
```

Scenario 2: A temporary shock to TFP

- The economy starts from the steady state
- There is an unexpected negative shock at the beginning of period 1: $\varepsilon_1 = -0.1$

Fragment from `rbc_det2.mod`

```
...
steady;

shocks;
  var epsilon;
  periods 1;
  values -0.1;
end;

perfect_foresight_setup(periods=300);
perfect_foresight_solver;
```

Scenario 3: Pre-announced favorable shocks in the future

- The economy starts from the steady state
- There is a sequence of positive shocks to A_t: 4% in period 5 and an additional 1% during the 4 following periods

Fragment from `rbc_det3.mod`

```
...
steady;

shocks;
  var epsilon;
  periods 4, 5:8;
  values 0.04, 0.01;
end;

perfect_foresight_setup(periods=300);
perfect_foresight_solver;
```

Scenario 4: A permanent shock

- The economy starts from the initial steady state ($a_0 = 0$)
- In period 1, TFP increases by 5% permanently (and this was unexpected)

Fragment from `rbc_det4.mod`

```
...
initval;
  epsilon = 0;
end;

steady;

endval;
  epsilon = (1-rho)*log(1.05);
end;

steady;
```

Scenario 5: A pre-announced permanent shock

- The economy starts from the initial steady state ($a_0 = 0$)
- In period 6, TFP increases by 5% permanently
- A `shocks` block is used to maintain TFP at its initial level during periods 1–5

Fragment from `rbc_det5.mod`

```
...
// Same initval and endval blocks as in Scenario 4
...

shocks;
  var epsilon;
  periods 1:5;
  values 0;
end;
```

Summary of commands

`initval`	for the initial steady state (followed by `steady`)
`endval`	for the terminal steady state (followed by `steady`)
`histval`	for initial or terminal conditions out of steady state
`shocks`	for shocks along the simulation path
`perfect_foresight_setup`	prepare the simulation
`perfect_foresight_solver`	compute the simulation

Under the hood

- The paths for exogenous and endogenous variables are stored in two MATLAB/Octave matrices:

$$\begin{array}{lcl} \texttt{oo\_.endo\_simul} & = & \begin{pmatrix} y_0 & y_1 & \dots & y_T & y_{T+1} \end{pmatrix} \\ \texttt{oo\_.exo\_simul'} & = & \begin{pmatrix} \boxtimes & u_1 & \dots & u_T & \boxtimes \end{pmatrix} \end{array}$$

- `perfect_foresight_setup` initializes those matrices, given the `shocks`, `initval`, `endval` and `histval` blocks
  - y_0, y_{T+1} and $u_1 \dots u_T$ are the constraints of the problem
  - $y_1 \dots y_T$ are the initial guess for the Newton algorithm
- `perfect_foresight_solver` replaces $y_1 \dots y_T$ in `oo_.endo_simul` by the solution
- Notes:
  - for historical reasons, dates are in columns in `oo_.endo_simul` and in lines in `oo_.exo_simul`, hence the transpose (`'`) above
  - this is the setup for no lead and no lag on exogenous
  - if one lead and/or one lag, u_0 and/or u_{T+1} would become relevant
  - if more than one lead and/or lag, matrices would be larger

Initial guess

The Newton algorithm needs an initial guess $Y^{(0)} = [y_1^{(0)\prime} \; \dots \; y_T^{(0)\prime}]$.

What is Dynare using for this?

- By default, if there is no `endval` block, it is the steady state as specified by `initval` (repeated for all simulations periods)
- Or, if there is an `endval` block, then it is the final steady state declared within this block
- Possibility of customizing this default by manipulating `oo_.endo_simul` after `perfect_foresight_setup` (but of course before `perfect_foresight_solver`!)
- If homotopy is triggered, the initial guess of subsequent iterations is the result of the previous iteration

Alternative way of specifying terminal conditions

- With the `differentiate_forward_vars` option of the `model` block, Dynare will substitute forward variables using new auxiliary variables:
  - Substitution: $x_{t+1} \to x_t + a_{t+1}$
  - New equation: $a_t = x_{t+1} - x_t$
- If the terminal condition is a steady state, the new auxiliary variables have obvious *zero* terminal condition
- Useful when:
  - the final steady state is hard to compute (this transformation actually provides a way to find it)
  - the model is very persistent and takes time to go back to steady state (this transformation avoids a kink at the end of the simulation if T is not large enough)

Outline

Zero nominal interest rate lower bound

- Implemented by writing the law of motion under the following form in Dynare:

$$i_t = \max\left\{0, (1-\rho_i)i^* + \rho_i i_{t-1} + \rho_\pi(\pi_t - \pi^*) + \varepsilon_t^i\right\}$$

- *Warning:* this form will be accepted in a stochastic model, but the constraint will not be enforced in that case!

Irreversible investment

Same model as above, but the social planner is constrained to positive investment paths:

$$\max_{\{c_{t+j},\ell_{t+j},k_{t+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} \beta^j u(c_{t+j},\ell_{t+j})$$

s.t.

$$y_t = c_t + i_t$$
$$y_t = A_t f(k_{t-1},\ell_t)$$
$$k_t = i_t + (1-\delta)k_{t-1}$$
$$i_t \geq 0$$
$$A_t = A^\star e^{a_t}$$
$$a_t = \rho\, a_{t-1} + \varepsilon_t$$

where the technology (f) and the preferences (u) are as above.

First order conditions

$$u_c(c_t,\ell_t) - \mu_t = \beta\,\mathbb{E}_t\left[u_c(c_{t+1},\ell_{t+1})\left(A_{t+1}f_k(k_t,\ell_{t+1}) + 1 - \delta\right) - \mu_{t+1}(1-\delta)\right]$$

$$\frac{u_\ell(c_t,\ell_t)}{u_c(c_t,\ell_t)} + A_t f_l(k_{t-1},\ell_t) = 0$$

$$c_t + k_t = A_t f(k_{t-1},\ell_t) + (1-\delta)k_{t-1}$$

Slackness condition:

$$\mu_t = 0 \ \text{ and } \ i_t \geq 0$$

or

$$\mu_t > 0 \ \text{ and } \ i_t = 0$$

where $\mu_t \geq 0$ is the Lagrange multiplier associated to the non-negativity constraint for investment.

Mixed complementarity problems

- A mixed complementarity problem (MCP) is given by:
  - function $F(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$
  - lower bounds $\ell_i \in \mathbb{R} \cup \{-\infty\}$
  - upper bounds $u_i \in \mathbb{R} \cup \{+\infty\}$
- A solution of the MCP is a vector $x \in \mathbb{R}^n$ such that for each $i \in \{1 \dots n\}$, one of the following alternatives holds:
  - $\ell_i < x_i < u_i$ and $F_i(x) = 0$
  - $x_i = \ell_i$ and $F_i(x) \geq 0$
  - $x_i = u_i$ and $F_i(x) \leq 0$
- Notation:

$$\ell \leq x \leq u \perp F(x)$$

- Solving a square system of nonlinear equations is a particular case (with $\ell_i = -\infty$ and $u_i = +\infty$ for all i)
- Optimality problems with inequality constraints are naturally expressed as MCPs (finite bounds are imposed on Lagrange multipliers)

The irreversible investment model in Dynare

- MCP solver triggered with option `lmmcp` of `perfect_foresight_solver`
- Slackness condition described by equation tag `mcp`

Fragment from `rbcii.mod`

```
  (c^theta*(1-L)^(1-theta))^(1-tau)/c - mu =
     beta*((c(+1)^theta*(1-L(+1))^(1-theta))^(1-tau)/c(+1)
      *(alpha*(y(+1)/k)^(1-psi)+1-delta)-mu(+1)*(1-delta));
...
  [ mcp = 'i > 0' ]
  mu = 0;
...
perfect_foresight_setup(periods=400);
perfect_foresight_solver(lmmcp, maxit=200);
```

OccBin (1/2)

- Piecewise linear approach of Guerrieri and Iacoviello (JME, 2015)
- Under certainty equivalence; but quite fast, works on large models
- Relies on a solution under perturbation and not a deterministic solver, hence nonlinearities other than regime change are not taken into account

Example

```
model;
  [name='Notional rate Taylor rule']
  i_not=rho*i_not(-1)+rho*(phi_pi*pie+phi_y*y)+zeps_i;
  [name='Observed interest rate', relax='zlb']
  i = i_not;
  [name='Observed interest rate', bind='zlb']
  i = i_elb;
...
end;
```

OccBin (2/2)

Example (cont'd)

```
occbin_constraints;
  name 'zlb'; bind i_not <= i_elb;
end;

shocks(surprise);
  var zeps_i;
  periods 1 2;
  values -0.01 -0.02;
end;

occbin_setup;
occbin_solver(simul_periods=20, simul_check_ahead_periods=50);
occbin_graph y i i_not pie;
```

Outline

Simulating unexpected shocks

With a perfect foresight solver:

- shocks are unexpected in period 1
- but in subsequent periods they are anticipated

How to simulate an unexpected shock at a period $t > 1$?

- Do a perfect foresight simulation from periods 0 to T *without the shock*
- Do another perfect foresight simulation from periods t to T
  - applying the shock in t,
  - and using the results of the first simulation as initial condition
- Combine the two simulations:
  - use the first one for periods 1 to $t - 1$,
  - and the second one for t to T

A Dynare example

Simulation of a scenario with:

- Pre-announced (negative) shocks in periods 5 and 15
- Unexpected (positive) shock in period 10

From `rbc_unexpected.mod`:

```
...
// Declare pre-announced shocks
shocks;
  var epsilon;
  periods 5, 15;
  values -0.1, -0.1;
end;

perfect_foresight_setup(periods=300);
perfect_foresight_solver;
```

A Dynare example (continued)

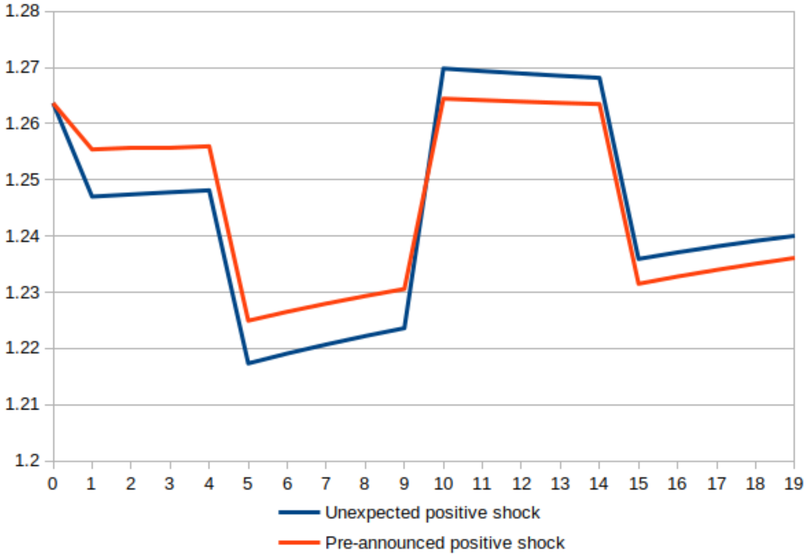
```
// Declare unexpected shock (after first simulation!)
oo_.exo_simul(11, 1) = 0.1; // Period 10 has index 11!

// Strip first 9 periods and save them
saved_endo = oo_.endo_simul(:, 1:9);  // Save periods 0 to 8
saved_exo = oo_.exo_simul(1:9, :);
oo_.endo_simul = oo_.endo_simul(:, 10:end); // Keep periods 9 to 301
oo_.exo_simul = oo_.exo_simul(10:end, :);

periods 291;
perfect_foresight_solver;

// Combine the two simulations
oo_.endo_simul = [ saved_endo oo_.endo_simul ];
oo_.exo_simul = [ saved_exo; oo_.exo_simul ];
```

Consumption path



Simplified syntax with Dynare 6

With the (future) Dynare 6, it will be possible to achieve the same with a simplified syntax.
From `rbc_unexpected_v6.mod`:

```
// Declare pre-announced shocks
shocks(learnt_in=1);
  var epsilon;
  periods 5, 15;
  values -0.1, -0.1;
end;

// Declare shocks learnt in period 10
shocks(learnt_in=10);
  var epsilon;
  periods 10;
  values 0.1;
end;
```

Simplified syntax with Dynare 6 (continued)

Then the full simulation (equivalent to two consecutive perfect foresight simulations) can be run with:

```
perfect_foresight_with_expectation_errors_setup(periods=300);
perfect_foresight_with_expectation_errors_solver;
```

Note:

- More complex scenarios are possible, where agents learn in period $t > 1$ about shock(s) in periods $t + s$ for $s > 0$ (*i.e.* pre-announced shocks, but which are learnt in some period > 1)
- Information can also be learnt about terminal conditions, using the `endval(learnt_in=...)` block

Outline

Extended path (EP) algorithm

- Idea: use the previous method to simulate a *rational expectations* (RE) model (*i.e.* with stochastic shocks that can happen at every period). . .
- . . . but under the simplifying assumption that agents believe that the economy will not be perturbed in the future (all future shocks will be at their steady state value $\bar{u} = 0$), and do not update their belief when observing shocks (hence we are not solving the true RE model, but an approximation of it)
- Advantage: *deterministic* nonlinearities fully taken into account
- Inconvenient: solution under certainty equivalence (Jensen inequality is violated). For example, no precautionary motive.
- When the goal is to generate a timeseries, this method is strictly superior to first-order perturbation (which is also under certainty equivalence, but does not take into account deterministic nonlinearities)
- Method introduced by Fair and Taylor (1983)

Extended path (EP) algorithm (continued)

- Algorithm

```
1 $H \leftarrow$ Set the horizon of the perfect foresight (PF) model
2 $(\bar{y}, \bar{u}) \leftarrow$ Compute steady state of the model
3 $y_0 \leftarrow$ Choose an initial condition for the endogenous variables
4 for $t = 1$ to $T$
5     $u_t \leftarrow$ Draw random shocks for the current period
6     $y_t \leftarrow$ Solve a PF with:
          ★ Initial condition: $y_{t-1}$ computed in previous iteration
          ★ Terminal condition: $y_{t+H} = \bar{y}$
          ★ Shocks: $u_t$ just drawn and $u_{t+s} = \bar{u}$ (for $s > 0$)
7 end for
```

- Implemented under the command `extended_path` (with option `order=0`, which is the default)
- Option `periods` controls T
- Option `solver_periods` controls H (defaults to 200)

Extended path: Dynare example

From `rbc_ep.mod`:

```
...
// Declare shocks as in a stochastic setup
shocks;
  var epsilon;
  stderr 0.02;
end;

extended_path(periods=300);

// Plot 20 first periods of consumption
ic = varlist_indices('c', M_.endo_names);
plot(oo_.endo_simul(ic, 1:21));
```

Stochastic extended path (SEP)

- Idea: generalize the extended path method to take into account *some* future uncertainty
- Approximation: at date t,
  - agents know that there will be future stochastic shocks in periods $t+1$ to $t+k$
  - but they assume that there will be no more shocks in periods $> t+k$
- k measures the degree of future uncertainty taken into account
- We are still not solving the true RE model, but the larger k, the closer we are to the true model (which corresponds to $k = +\infty$).
- Note: (plain) extended path as presented in the previous slides corresponds to $k = 0$
- Additional approximation: the probability distribution about future uncertainty is simplified using discrete numerical integration (*a.k.a.* quadrature)

Gauss-Hermite quadrature (univariate)

- Let X be a Gaussian random variable with mean zero and variance $\sigma_x^2 > 0$, and suppose that we need to evaluate $\mathbb{E}[\varphi(X)]$, where φ is a continuous function
- By definition we have:

$$\mathbb{E}[\varphi(X)] = \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{-\frac{x^2}{2\sigma_x^2}} \mathrm{d}x$$

- This integral can be approximated by a finite sum using the following result (Gauss-Hermite quadrature formula at order n):

$$\int_{-\infty}^{\infty} \varphi(z) e^{-z^2} \mathrm{d}z \approx \sum_{i=1}^{n} \omega_i \varphi(z_i)$$

  where z_i $(i = 1, \dots, n)$ are the roots of an order n Hermite polynomial, and the weights ω_i are positive and summing up to one (variable change: $x_i = \frac{z_i}{\sigma_x\sqrt{2}}$)
- The higher n, the better the approximation

Gauss-Hermite quadrature (multivariate)

- Let X be a multivariate Gaussian random variable with mean zero and unit variance, and suppose that we need to evaluate

$$\mathbb{E}[\varphi(X)] = (2\pi)^{-\frac{p}{2}} \int_{\mathbb{R}^p} \varphi(\mathbf{x}) e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}} \mathrm{d}\mathbf{x}$$

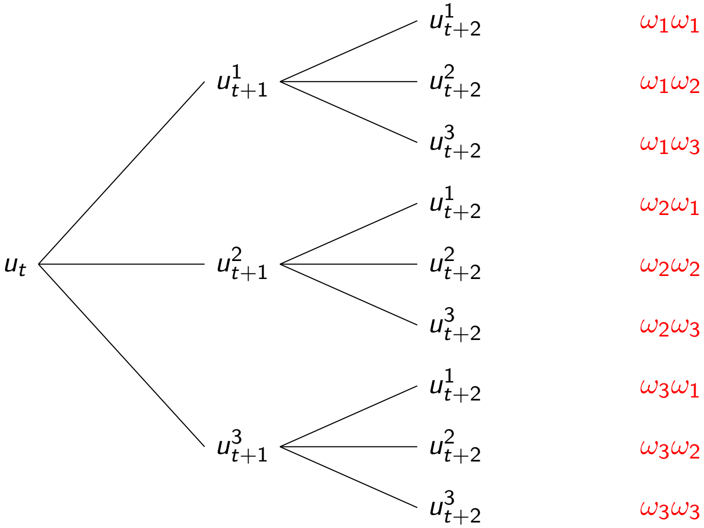
- Let $\{(\omega_i, z_i)\}_{i=1}^n$ be the weights and nodes of an order n univariate Gauss-Hermite quadrature
- This integral can be approximated using a tensor grid:

$$\int_{\mathbb{R}^p} \varphi(\mathbf{z}) e^{-\mathbf{z}'\mathbf{z}} \mathrm{d}\mathbf{z} \approx \sum_{i_1,\dots,i_p=1}^{n} \omega_{i_1} \dots \omega_{i_p} \varphi(z_{i_1}, \dots, z_{i_p})$$

- **Curse of dimensionality:** The number of terms in the sum grows exponentially with the number of shocks.

Forward history

One shock, three quadrature nodes, order two SEP ($k = 2$)



Stochastic extended path (SEP) (continued)

- Algorithm similar to extended path (EP), except that instead of solving a perfect foresight (PF) problem for each period, a larger problem is solved:
  - equations determining future variables are replicated as many times as there are branches on the tree of history;
  - Gauss-Hermite quadratures are used to compute expectations against those future variables.
- We face two curses of dimensionality (exponential complexity growth):
  - Stochastic order (k)
  - Number of shocks
- In practice, only feasible for small k and small number of shocks
- SEP triggered with option `order=k` of `extended_path` command
- NB: currently no interface for controlling the number of nodes for the Gauss-Hermite quadrature. The number of nodes has to be directly set in the `options_` global structure

Outline

Local approximation of stochastic models

The general problem:

$$\mathbb{E}_t f(y_{t+1}, y_t, y_{t-1}, u_t) = 0$$

y : vector of endogenous variables

u : vector of exogenous shocks

with:

$$\begin{aligned} \mathbb{E}(u_t) &= 0 \\ \mathbb{E}(u_t u_t') &= \Sigma_u \\ \mathbb{E}(u_t u_s') &= 0 \text{ for } t \neq s \end{aligned}$$

What is a solution to this problem?

- A solution is a policy function of the form:

$$y_t = g\left(y_{t-1}, u_t, \sigma\right)$$

where σ is the *stochastic scale* of the problem and:

$$u_{t+1} = \sigma\, \varepsilon_{t+1}$$

- The policy function must satisfy:

$$\mathbb{E}_t f\left(g\left(g\left(y_{t-1}, u_t, \sigma\right), u_{t+1}, \sigma\right), g\left(y_{t-1}, u_t, \sigma\right), y_{t-1}, u_t\right) = 0$$

Local approximations

$$
\begin{aligned}
\hat{g}^{(1)}\left(y_{t+1}, u_t, \sigma\right) &= \bar{y} + g_y \hat{y}_{t-1} + g_u u_t \\
\hat{g}^{(2)}\left(y_{t+1}, u_t, \sigma\right) &= \bar{y} + \frac{1}{2} g_{\sigma\sigma} + g_y \hat{y}_{t-1} + g_u u_t \\
&\quad + \frac{1}{2}\left(g_{yy}\left(\hat{y}_{t-1} \otimes \hat{y}_{t-1}\right) + g_{uu}\left(u_t \otimes u_t\right)\right) \\
&\quad + g_{yu}\left(\hat{y}_{t-1} \otimes u_t\right) \\
\hat{g}^{(3)}\left(y_{t+1}, u_t, \sigma\right) &= \bar{y} + \frac{1}{2} g_{\sigma\sigma} + \frac{1}{6} g_{\sigma\sigma\sigma} + \frac{1}{2} g_{\sigma\sigma y} \hat{y}_{t-1} + \frac{1}{2} g_{\sigma\sigma u} u_t \\
&\quad + g_y \hat{y}_{t-1} + g_u u_t + \ldots
\end{aligned}
$$

Breaking certainty equivalence (1/2)

The combination of future uncertainty (future shocks) and nonlinear relationships makes for precautionary motives or risk premia.

- 1st order: certainty equivalence; today's decisions don't depend on future uncertainty
- 2nd order:

$$\begin{aligned}\hat{g}^{(2)}\left(y_{t+1}, u_t, \sigma\right) = \bar{y} &+ \frac{1}{2} g_{\sigma\sigma} + g_y \hat{y}_{t-1} + g_u u_t \\ &+ \frac{1}{2}\left(g_{yy}\left(\hat{y}_{t-1} \otimes \hat{y}_{t-1}\right) + g_{uu}\left(u_t \otimes u_t\right)\right) \\ &+ g_{yu}\left(\hat{y}_{t-1} \otimes u_t\right)\end{aligned}$$

Risk premium is a constant: $\frac{1}{2} g_{\sigma\sigma}$

Breaking certainty equivalence (2/2)

- 3rd order:

$$\hat{g}^{(3)}\left(y_{t+1}, u_t, \sigma\right) = \bar{y} + \frac{1}{2} g_{\sigma\sigma} + \frac{1}{6} g_{\sigma\sigma\sigma} + \frac{1}{2} g_{\sigma\sigma y} \hat{y}_{t-1} + \frac{1}{2} g_{\sigma\sigma u} u_t + g_y \hat{y}_{t-1} + g_u u_t + \dots$$

Risk premium is linear in the state variables:

$$\frac{1}{2} g_{\sigma\sigma} + \frac{1}{6} g_{\sigma\sigma\sigma} + \frac{1}{2} g_{\sigma\sigma y} \hat{y}_{t-1} + \frac{1}{2} g_{\sigma\sigma u} u_t$$

The cost of local approximations

1. High order approximations are accurate around the steady state, and more so than lower order approximations
2. But can be totally wrong far from the steady state (and may be more so than lower order approximations)
3. Error of approximation of a solution $\hat{g}$, at a given point of the state space (y_{t-1}, u_t):

$$\mathcal{E}\left(y_{t-1}, u_t\right) = \mathbb{E}_t f\left(\hat{g}\left(\hat{g}\left(y_{t-1}, u_t, \sigma\right), u_{t+1}, \sigma\right), \hat{g}\left(y_{t-1}, u_t, \sigma\right), y_{t-1}, u_t\right)$$

4. Necessity for pruning

Approximation of occasionally binding constraints with penalty functions

The investment positivity constraint is translated into a penalty on the welfare:

$$\max_{\{c_{t+j},\ell_{t+j},k_{t+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} \beta^j u(c_{t+j},\ell_{t+j}) + h \cdot \log(i_{t+j})$$

s.t.

$$y_t = c_t + i_t$$
$$y_t = A_t f(k_{t-1}, \ell_t)$$
$$k_t = i_t + (1-\delta)k_{t-1}$$
$$A_t = A^\star e^{a_t}$$
$$a_t = \rho\, a_{t-1} + \varepsilon_t$$

where the technology (f) and the preferences (u) are as before, and h governs the strength of the penalty (*barrier parameter*)

Thanks for your attention!
Questions?

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